International Journal of Geometric Methods in Modern Physics

A GEOMETRIC METHOD OF CONSTRUCTING EXACT SOLUTIONS IN MODIFIED f(R,T) GRAVITY WITH YANG–MILLS AND HIGGS INTERACTIONS

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Received (Day Month Year)
Revised (Day Month Year)

We show that a geometric techniques can be elaborated and applied for constructing generic off-diagonal exact solutions in $f(R,T)$–modified gravity for systems of gravitational–Yang-Mills–Higgs equations. The corresponding classes of metrics and generalized connections are determined by generating and integration functions which depend, in general, on all space and time coordinates and may possess, or not, Killing symmetries. For nonholonomic constraints resulting in Levi–Civita configurations, we can extract solutions of the Einstein–Yang–Mills–Higgs equations. We show that the constructions simplify substantially for metrics with at least one Killing vector. There are provided and analyzed some examples of exact solutions describing generic off–diagonal modifications to black hole/ellipsoid and solitonic configurations.

MSC 2010: 83T13, 83C15 (primary); 53C07, 70S15 (secondary)
PACS 2008: 04.50.Kd, 04.90.+e, 11.15.Kc

Keywords: Modified gravity, EYMH equations, exact solutions, black holes, gravitational solitons

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1. Introduction

The focus of this paper is the question if the fundamental gravitational field and matter fields equations in theories of modified gravity can be integrated in certain general forms with generic off–diagonal metrics and (generalized) connections depending on all spacetime coordinates. Recently, a number of modified gravity theories have been elaborated with the aim to explain the acceleration [1,2] of universe and in various attempts to formulate renormalizable quantum gravity models; for reviews on $f(R)$–gravity and related theories, see [3,4,5,6,7,8]. The corresponding classical field equations for the Einstein gravity and generalized/modified gravity theories consist very sophisticate systems of nonlinear partial differential equations (PDE).

The aim of this work is to present a treatment of modified gravitational field and Yang–Mills–Higgs matter field equations, MGYMH, and developing new geometric methods for integrating such PDE. Such solutions and methods are not presented in well–known monographs on exact solutions in gravity [9,10] and in Ref. [11]. We shall apply the so–called anholonomic frame deformation method, AFDM, see [12,13,14,15] and references therein, for constructing exact solutions of PDE in mathematical (modified) particle physics describing generic off–diagonal and nonlinear gravitational, gauge and scalar field interactions. A series of examples of off–diagonal solutions for MGYMH black holes/ellpsoids and solitonic configurations will be analyzed.

The main idea of the AFDM is to work with an auxiliary linear connection $\mathbf{D}$, which is necessary for decoupling the (modified) gravitational field and matter field equations with respect to certain classes of adapted frames with 2+2 splitting. This allows us to integrate physically important PDE in very general off–diagonal forms. It is not possible to decouple such PDE for the Levi–Civita connection, ∇, and/or in arbitrary coordinate frames. Nevertheless, we can extract exact and approximate solutions for ∇ by imposing nonholonomic constraints on $\mathbf{D}$ and corresponding generalized Ricci tensor, $\widehat{R}_{\alpha\beta}$, and scalar curvature, ${}^{s}R$. In this this approach, the $f({}^{s}R, T)$–modified gravity and related systems of gravitational–Yang-Mills–Higgs (MGYMH) equations are determined by a functional $f({}^{s}R, T)$, where T is the trace of the energy–momentum for matter fields. If $\mathbf{D} = \nabla$ and $f({}^{s}R, T) = R$ for the corresponding Levi–Civita connection, ∇, and corresponding curvature scalar, R, we extract solutions of the Einstein–Yang–Mills–Higgs (EYMH) equations. We shall demonstrate that a number of physically important effects in modified gravity theories can be equivalently modelled by off–diagonal interactions (with certain nonholonomic constraints) in GR.

We shall proceed as follows. In Section 2.1 we provide geometric preliminaries and prove the main Theorems on decoupling the modified gravitational equations for generic off–diagonal metrics with one Killing symmetry. There are considered also extensions to classes of "non–Killing" solutions with coefficients depending on all set of four coordinates. We show that the decoupling property also holds true for certain classes of nonholonomic MGYMH systems. The theorems on generating off–diagonal solutions for effectively polarized cosmological constants are considered in Section 3. Next two sections are devoted to examples of generic off–diagonal exact solutions for MGYMH systems. In Section 4, we study the geometry of nonholonomic YMH vacuum deformations of black holes. Certain examples of ellipsoid–solitonic non–Abbelian configurations and related modifications are analyzed in Section 5. Appendix Appendix A contains some most important component formulas in the geometry of nonholonomic manifolds 2+2 splitting. A proof of the decoupling property is given in Appendix Appendix B.

2. Nonholonomic Splitting in Modified Gravity

2.1. *Geometric preliminaries*

We consider modified gravity theories, when the geometric models of curved spacetime $(V, \mathbf{g}, \mathbf{D})$ are determined by a generalized pseudo–Riemannian manifold V endowed with a (modified) Lorentzian metric $\mathbf{g}$ and a metric compatible linear connection $\mathbf{D}$, when $\mathbf{D}\mathbf{g} = 0$. The geometric/physical data $(\mathbf{g}, \mathbf{D})$ are supposed to define solutions of certain systems of gravitational field equations, see below section 2.2, and $(\mathbf{g}, \mathbf{D} = \nabla)$ as solutions of the Einstein equations.[a]

2.1.1. *Metrics adapted to nonholonomic 2+2 splitting*

On a coordinate neighborhood $U \subset V$, the local coordinates $u = \{u^\alpha = (x^i, y^a)\}$ are considered with conventional $2+2$ splitting into h-coordinates, $x = (x^i)$, and v-coordinates, $y = (y^a)$, for $j, k, ... = 1, 2$ and $a, b, c, ... = 3, 4$. The local coordinate basis and cobases are respectively $e_\alpha = \partial_\alpha = \partial/\partial u^\beta$ and $e^\beta = du^\beta$, which can transformed into arbitrary frames (tetrads/vierbeinds) via transforms of type $e_{\alpha'} = e^{\alpha}_{\ \alpha'}(u) e_\alpha$ and $e^{\alpha'} = e_{\alpha}^{\ \alpha'}(u) e^\alpha$.[b] The coefficients of a vector X and a metric $\mathbf{g}$ are defined, respectively, in the forms $X = X^\alpha e_\alpha$ and

$$\mathbf{g} = g_{\alpha\beta}(u) e^\alpha \otimes e^\beta, \tag{1}$$

where $g_{\alpha\beta} := \mathbf{g}(e_\alpha, e_\beta)$. For our purposes, we shall work with bases with non–integrable (equivalently, nonholonomic/anholonomic) h–v–decomposition, when for the tangent bundle $TV := \bigcup_u T_u V$ a Whitney sum

$$\mathbf{N} : \ TV = hV \oplus vV \tag{2}$$

is globally defined. In local form, this is defined by a nonholonomic distribution with coefficients $N_i^a(u)$, when $\mathbf{N} = N_i^a(x, y) dx^i \otimes \partial/\partial y^a$.[c]

[a]We suppose that readers are familiar with basic concepts from differential geometry and physical mathematics.

[b]The summation rule on repeating low–up indices will be applied if the contrary will be not stated.

[c]For varios theories of gravity and different variables in corresponding geometric models, a h–v–splitting defines a nonlinear connection, N–connection, structures. On (pseudo) Riemannian

It is possible to adapt the geometric constructions to a N–splitting (2) if we work with "N–elongated" local bases (partial derivatives), $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$, and cobases (differentials), $\mathbf{e}^\mu = (e^i, \mathbf{e}^a)$, when

$$\mathbf{e}_i = \partial/\partial x^i - \ N_i^a(u)\partial/\partial y^a, \ e_a = \partial_a = \partial/\partial y^a, \tag{3}$$

$$\text{and } e^i = dx^i, \ \mathbf{e}^a = dy^a + \ N_i^a(u)dx^i. \tag{4}$$

For instance, the basic vectors (3) satisfy the nonholonomy relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W^\gamma_{\alpha\beta}\mathbf{e}_\gamma, \tag{5}$$

with (antisymmetric) nontrivial anholonomy coefficients

$$W^b_{ia} = \partial_a N_i^b, W^a_{ji} = \Omega^a_{ij} = \mathbf{e}_j\left(N_i^a\right) - \mathbf{e}_i(N_j^a). \tag{6}$$

The "boldface" letters are used in order to emphasize that a spacetime model $(\mathbf{V}, \mathbf{g}, \mathbf{D})$ and certain geometric objects and constructions are "N–adapted", i. e. adapted to a h–v–splitting. The geometric objects are called distinguished (in brief, d–objects, d–vectors, d–tensors etc) if they are adapted to the N–connection structure via decompositions with respect to frames of type (3) and (4). For instance, we write a d–vector as $\mathbf{X} = (hX, vX)$ and a d–metric as $\mathbf{g} = (hg, vg)$.

Proposition 1. ***-Definition.*** *Any spacetime metric* $\mathbf{g}$ *(1) can be represented equivalently as a d–metric,*

$$\mathbf{g} = \ g_{ij}(x,y) \ e^i \otimes e^j + \ g_{ab}(x,y) \ \mathbf{e}^a \otimes \mathbf{e}^b, \tag{7}$$

for $hg = \{\ g_{ij}\}$ *and* $vg = \{g_{ab}\}$.

Proof. Via frame transforms, $e_\alpha = e^{\alpha'}_{\ \alpha}(x,y)e_{\alpha'}$, $\underline{g}_{\alpha\beta} = e^{\alpha'}_{\ \alpha} e^{\beta'}_{\ \beta} \ \underline{g}_{\alpha'\beta'}$, any metric $\mathbf{g} = \underline{g}_{\alpha\beta} du^\alpha \otimes du^\beta$ can be parameterized in the form

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b g_{ab} & N_j^e g_{ae} \\ N_i^e g_{be} & g_{ab} \end{bmatrix} \tag{8}$$

for any prescribed set of coefficients N_i^a. A metric $\mathbf{g}$ is generic off–diagonal if the matrix (8) can not be diagonalized via coordinate transforms. For such a metric, the anholonomy coefficients (6) are not zero. Regrouping the terms for co–bases (4), we obtain the d–metric (7). Parameterizations of this type are used in Kaluza–Klein gravity when $N_i^a(x,y) = \Gamma^a_{bi}(x)y^a$ and y^a are "compactified" extra–dimensions coordinates, or in Finsler gravity theories, see details in [14]. In this work, we restrict our considerations only to four dimensional (4–d) gravity theories. □ □

We shall study exact solutions with metrics which via frame/ coordinate transforms can be related to a d–metric $\mathbf{g}$ (7) and/or ansatz (8) and written in a form

manifolds to consider such a N–connection is equivalent to a prescription of class of N–elongated frames.

with separation of v–coordinates, y^3 and y^4, and nontrivial vertical conformal transforms,

$$\mathbf{g} = g_i dx^i \otimes dx^i + \omega^2 h_a \underline{h}_a \mathbf{e}^a \otimes \mathbf{e}^a, \quad (9)$$
$$\mathbf{e}^3 = dy^3 + (w_i + \underline{w}_i)\, dx^i,\ \mathbf{e}^4 = dy^4 + (n_i + \underline{n}_i)\, dx^i,$$

where

$$g_i = g_i(x^k), g_a = \omega^2(x^i, y^c)\ h_a(x^k, y^3)\underline{h}_a(x^k, y^4),$$
$$N_i^3 = w_i(x^k, y^3) + \underline{w}_i(x^k, y^4), N_i^4 = n_i(x^k, y^3) + \underline{n}_i(x^k, y^4), \quad (10)$$

are functions of necessary smooth class which will be defined in a form to generate solutions of certain fundamental gravitational and matter field equations.[d]

2.1.2. *N–adapted connections*

Linear connection structures can be introduced on a generalized spacetime $\mathbf{V}$ in form which is N–adapted to a h–v–splitting (2), or not.

Definition 2. *A d–connection* $\mathbf{D} = (hD, vD)$ *is a linear connection preserving under parallelism the N–connection structure.*

Any d–connection defines a covariant N–adapted derivative $\mathbf{D_X Y}$ of a d–vector field $\mathbf{Y}$ in the direction of a d–vector $\mathbf{X}$. With respect to N–adapted frames (3) and (4), any $\mathbf{D_X Y}$ can be computed as in GR but with the coefficients of the Levi–Civita connection substituted by $\mathbf{D} = \{\mathbf{\Gamma}^\gamma_{\ \alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})\}$.

Any d–connection is characterized by three fundamental geometric objects: the d–torsion field which is (by definition)

$$\mathcal{T}(\mathbf{X}, \mathbf{Y}) := \mathbf{D_X Y} - \mathbf{D_Y X} - [\mathbf{X}, \mathbf{Y}];$$

the d–curvature field,

$$\mathcal{R}(\mathbf{X}, \mathbf{Y}) := \mathbf{D_X D_Y} - \mathbf{D_Y D_X} - \mathbf{D}_{[\mathbf{X},\mathbf{Y}]};$$

and the nonmetricity field is $\mathcal{Q}(\mathbf{X}) := \mathbf{D_X g}$. We compute the N–adapted coefficients of these geometric objects by introducing $\mathbf{X} = \mathbf{e}_\alpha$ and $\mathbf{Y} = \mathbf{e}_\beta$, defined by (3), and $\mathbf{D} = \{\mathbf{\Gamma}^\gamma_{\ \alpha\beta}\}$ into above formulas,

$$\mathcal{T} = \{\mathbf{T}^\gamma_{\ \alpha\beta} = \left(T^i_{\ jk}, T^i_{\ ja}, T^a_{\ ji}, T^a_{\ bi}, T^a_{\ bc}\right)\};$$
$$\mathcal{R} = \{\mathbf{R}^\alpha_{\ \beta\gamma\delta} = \left(R^i_{\ hjk}, R^a_{\ bjk}, R^i_{\ hja}, R^c_{\ bja}, R^i_{\ hba}, R^c_{\ bea}\right)\};\ \mathcal{Q} = \{\mathbf{Q}^\gamma_{\ \alpha\beta}\}.$$

[d]There is not summation on repeating "low" indices a in formulas (10) but such a summation is considered for crossing "up–low" indices i and a in (9)). We shall underline a function if it positively depends on y^4 but not on y^3 and write, for instance, $\underline{n}_i(x^k, y^4)$.

The formulas for $\mathbf{T}^{\gamma}{}_{\alpha\beta}$ and $\mathbf{R}^{\alpha}{}_{\beta\gamma\delta}$ are similar, respectively, to (A.3) and (A.4), see Appendix, but without "hats" on geometric objects.

Theorem 3. -***Definition.*** *There is a canonical d–connection* $\widehat{\mathbf{D}}$ *uniquely determined by any given* $\mathbf{g} = \{g_{\alpha\beta}\}$ *for a prescribed* $\mathbf{N} = \{N_i^a\}$ *which is metric compatible,* $\widehat{\mathbf{D}}\mathbf{g} = \mathbf{0}$, *and with zero h–torsion,* $h\widehat{\mathcal{T}} = \{\widehat{T}^i{}_{jk}\} = 0$, *and zero v–torsion,* $v\widehat{\mathcal{T}} = \{\widehat{T}^a{}_{bc}\} = 0$.

Proof. It follows from a straightforward verification in N–adapted frames that $\widehat{\mathbf{D}} = \{\ \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})\}$ with coefficients

$$\begin{aligned} \widehat{L}^i_{jk} &= \frac{1}{2} g^{ir} \left(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk} \right), \\ \widehat{L}^a_{bk} &= e_b(N^a_k) + \frac{1}{2} g^{ac} \left(\mathbf{e}_k g_{bc} - g_{dc}\ e_b N^d_k - g_{db}\ e_c N^d_k \right), \\ \widehat{C}^i_{jc} &= \frac{1}{2} g^{ik} e_c g_{jk},\ \widehat{C}^a_{bc} = \frac{1}{2} g^{ad} \left(e_c g_{bd} + e_b g_{cd} - e_d g_{bc} \right), \end{aligned} \tag{13}$$

computed for a d–metric $\mathbf{g} = [g_{ij}, g_{ab}]$ (7). □ □

Every metric field $\mathbf{g}$ naturally and completely defines a Levi–Civita connection $D = \nabla = \{\Gamma^{\gamma}{}_{\alpha\beta}\}$ if and only if there are satisfied the metric compatibility, ${}^{\nabla}\mathcal{Q}(X) = \nabla_{\mathbf{X}}\mathbf{g} = 0$, and zero torsion, ${}^{\nabla}\mathcal{T} = 0$, conditions.[e] For $\widehat{\mathbf{D}}$, we can find a unique distortion relation

$$\nabla = \widehat{\mathbf{D}} + \widehat{\mathbf{Z}}, \tag{14}$$

where both linear connections ∇ and $\widehat{\mathbf{D}}$ and the distortion tensor $\widehat{\mathbf{Z}}$ are completely defined by $\mathbf{g} = \{g_{\alpha\beta}\}$ for a prescribed $\mathbf{N} = \{N_i^a\}$, see details in [12,13,14] and Theorem 15. With respect to N–adapted frames, the formula (14) is given by (A.1) when the coefficients of the distortion tensor $\widehat{\mathbf{Z}}$ is determined by values (A.2). It is important to emphasize that such formulas define a unique deformation of the Christoffel symbols for ∇ into the corresponding coefficients (13) of $\widehat{\mathbf{D}}$ because all such values are completely defined by the coefficients of a metric tensor (8) (equivalently, (7)). In N–adapted form, the coefficients of the Levi–Civita connection (i.e. the second type Christoffel symbols) can be computed by taking respective sums of (13) and (A.2).

The curvature, Ricci and Einstein tensors of the connections ∇ and $\widehat{\mathbf{D}}$ are computed respectively by standard formulas (A.4), (A.5) and (A.7) (see details in Appendix Appendix A). For instance, the curvature ${}^{\nabla}\mathcal{R}$ for ∇ can be computed as a distortion of $\widehat{\mathcal{R}} = \{\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\delta}\}$ (A.4) for $\widehat{\mathbf{D}}$ using the distortion relation for connections (14) (and (A.1)). The properties of the Ricci tensor of $\widehat{\mathbf{D}}$ are stated by Proposition

[e]We emphasize that for geometric/physical objects defined by ∇ we do not use "boldface" symbols because this linear connection does not preserve under parallelism and general frame/coordinate transforms a N–splitting (2).

19, see also formulas (A.8) and (A.9). Here we note that, in general, the Ricci tensor $\widehat{\mathbf{R}}\ _{\beta\gamma}$ is not not symmetric because of nontrivial nonholonomically induced torsion $\widehat{\mathcal{T}}$. Nevertheless, such nonsymetric contributions do not result in nonsymmetric metrics if we consider nonholonomic deformations determined by distortions (14) computed by symmetric metrics when nonysmmetric components of $\widehat{\mathbf{R}}\ _{\beta\gamma}$ are zero.

We conclude that all geometric constructions and physical theories derived for the geometric data $(\mathbf{g},\nabla)$ can be equivalently modeled by the data $(\mathbf{g},\mathbf{N},\widehat{\mathbf{D}})$ because of a unique distortion relation (14). If we work with $\widehat{\mathbf{D}}$, we have non–trivial nonholonomically induced torsion coefficients (A.3). The meaning of such a torsion structure is different from that, for instance, in Riemann–Cartan geometry when certain additional spin like sources are considered for additional algebraic type field equations for torsion fields. In our approach, $\widehat{\mathcal{T}}$ is completely defined by the metric structure when a N–splitting is prescribed.

2.2. *Nonholonomic MG field equations*

We study modified gravity theories derived for the action

$$S=\frac{1}{16\pi}\int \delta u\sqrt{|\mathbf{g}_{\alpha\beta}|}[f(\ ^{s}\widehat{R},T)+\ ^{m}L], \tag{15}$$

where $\ ^{m}L$ is the matter Lagrangian density; the stress–energy tensor of matter is computed via variation on inverse metric tensor, $\ ^{m}\mathbf{T}_{\alpha\beta}=-\frac{2}{\sqrt{|\mathbf{g}_{\mu\nu}|}}\frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|}\ ^{m}L)}{\delta\mathbf{g}^{\alpha\beta}}$, trace $T:=\mathbf{g}^{\alpha\beta}\ ^{m}\mathbf{T}_{\alpha\beta}$, and $f(\ ^{s}\widehat{R},\ ^{m}T)$ is an arbitrary functional on $\ ^{s}\widehat{R}$ (A.6) and $\ ^{m}T$. The volume form δu is determined by a d–metric $\mathbf{g}$ (7) in order to derive variational formulas in N–adapted form. For simplicity, we can assume that in cosmological models the stress–energy tensor of the matter is given by

$$\ ^{m}\mathbf{T}_{\alpha\beta}=(\rho+p)\mathbf{v}_{\alpha}\mathbf{v}_{\beta}-p\mathbf{g}_{\alpha\beta}, \tag{16}$$

where in the approximation of perfect fluid matter ρ is the energy density, p is the pressure and the four–velocity $\mathbf{v}_{\alpha}$ is subjected to the conditions $\mathbf{v}_{\alpha}\mathbf{v}^{\alpha}=1$ and $\mathbf{v}^{\alpha}\widehat{\mathbf{D}}_{\beta}\mathbf{v}_{\alpha}=0$, for $\ ^{m}L=-p$ in a corresponding local frame. We also consider approximations of type

$$f(\ ^{s}\widehat{R},\ ^{m}T)=\ ^{1}f(\ ^{s}\widehat{R})+\ ^{2}f(\ ^{m}T) \tag{17}$$

and denote by $\ ^{1}F(\ ^{s}\widehat{R}):=\partial\ ^{1}f(\ ^{s}\widehat{R})/\partial\ ^{s}\widehat{R}$ and $\ ^{2}F(\ ^{m}T):=\partial\ ^{2}f(\ ^{m}T)/\partial\ \ ^{m}T.$

In this work, we consider effective sources parameterized with respect to N–adapted frames in the form

$$\Upsilon_{\beta\delta}=\ ^{ef}\eta\ G\ \ ^{m}\mathbf{T}_{\beta\delta}+\ ^{ef}\mathbf{T}_{\beta\delta}, \tag{18}$$

with effective polarization of cosmological constant $\ ^{ef}\eta=[1+\ ^{2}F/8\pi]/\ ^{1}F$ and where the f–modification of the energy–momentum tensor is computed as an additional effective source

$$\ ^{ef}\mathbf{T}_{\beta\delta}=[\frac{1}{2}(\ ^{1}f-\ ^{1}F\ ^{s}\widehat{R}+2p\ ^{2}F+\ ^{2}f)\mathbf{g}_{\beta\delta}-(\mathbf{g}_{\beta\delta}\ \widehat{\mathbf{D}}_{\alpha}\widehat{\mathbf{D}}^{\alpha}-\widehat{\mathbf{D}}_{\beta}\widehat{\mathbf{D}}_{\delta})\ ^{1}F]/\ ^{1}F. \tag{19}$$

We can postulate in geometric form or prove following a variational approach:

Theorem 4. *The gravitational field equations for a modified gravity model (15) with f–functional (17) and perfect fluid stress–energy tensor (16) can be re–written equivalently using the canonical d–connection $\widehat{\mathbf{D}}$,*

$$\widehat{\mathbf{R}}_{\ \beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta}\ {}^{s}R = \mathbf{\Upsilon}_{\beta\delta}, \tag{20}$$

where the source d–tensor $\mathbf{\Upsilon}_{\beta\delta}$ is such way constructed that $\mathbf{\Upsilon}_{\beta\delta} \to 8\pi G T_{\beta\delta}$ for $\widehat{\mathbf{D}} \to \nabla$, where $T_{\beta\delta}$ is the energy–momentum tensor in GR with coupling gravitational constant G.

Proof. By varying the action S (15) with respect to $\mathbf{g}^{\alpha\beta}$ and following covariant differential calculus with N–elongated operators (3) and (4), we obtain the gravitational field equations (20) and respective effective sources (18) and (19). We omit technical details of such proofs because they are similar to the results for the Levi–Civita connection in [5] but (in our case) with distortions of formulas containing covariant derivatives to be defined by the canonical d–connection, see (14) and (A.1). All such constructions are metric compatible and determined by the same metric structure. For ${}^{ef}\mathbf{T}_{\beta\delta} = 0$, such details can be found in Refs. [12,13,14] Effective source from modified gravity do no change the N–adapted variational calcus.
□ □

We consider matter field sources in (20) which can be diagonalized with respect to N–adapted frames,

$$\mathbf{\Upsilon}^{\beta}_{\ \delta} = diag[\mathbf{\Upsilon}_{\alpha} : \mathbf{\Upsilon}^{1}_{\ 1} = \mathbf{\Upsilon}^{2}_{\ 2} = \Upsilon(x^{k}, y^{3}) + \underline{\Upsilon}(x^{k}, y^{4}); \mathbf{\Upsilon}^{3}_{\ 3} = \mathbf{\Upsilon}^{4}_{\ 4} = {}^{v}\Upsilon(x^{k})]. \tag{21}$$

It can be performed via frame/ coordinate transforms for very general distributions of matter fields. Such effective sources can be considered as nonholonomic constraints via corresponding classes of f–functionals (17) on the Ricci tensor (see Theorem 6) and certain classes of computed for certain general assumptions on modified off–diagonal gravitational interactions.

Corollary 5. *The gravitational field equations (20) transform into the Einstein equations in GR, in "standard" form for ∇,*

$$E_{\beta\delta} = R_{\beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta}\ R = \varkappa\ {}^{m}T_{\beta\delta}, \tag{22}$$

where $R := \mathbf{g}^{\beta\delta} R_{\ \beta\delta}$, if ${}^{2}f = 0$, and ${}^{1}f(\ {}^{s}\widehat{R}) = R$, for the same N–adapted coefficients for both $\widehat{\mathbf{D}}$ and ∇ if

$$\widehat{L}^{c}_{aj} = e_{a}(N^{c}_{j}),\ \widehat{C}^{i}_{jb} = 0,\ \Omega^{a}_{\ ji} = 0, \tag{23}$$

Proof. The systems of PDE (20) and (22) are very different. But if the constraints (23) are imposed additionally on $\widehat{\mathbf{D}}$, we satisfy the conditions of Proposition 19, when the equalities $\Gamma^{\gamma}_{\ \alpha\beta} = \widehat{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}$ result in $R_{\beta\delta} = \widehat{\mathbf{R}}_{\ \beta\delta}$ and $E_{\alpha\beta} = \widehat{\mathbf{E}}_{\alpha\beta}$. □ □

Finally, we note that the effective source $\mathbf{\Upsilon}^{\beta}{}_{\delta}$ and the canonical d–connection $\widehat{\mathbf{D}}$ encode all information on modifications of the GR theory. Prescribing such values following certain geometric/physical assumptions, we can model modified gravity effects via generic off–diagonal solutions in GR if we are able to construct such metrics via nonholonomic constraints $\widehat{\mathbf{D}}_{\mathcal{T}=0} \rightarrow \nabla$.

2.3. *Decoupling of modified gravitational field eqs*

2.3.1. *Off–diagonal metrics with Killing symmetry*

Let us consider an ansatz (9) with $\omega = 1, \underline{h}_3 = 1, \underline{w}_i = 0$ and $\underline{n}_i = 0$ for the data (10) and $\underline{\mathbf{\Upsilon}} = 0$ for (21). Such a generic off–diagonal metric does not depend on variable y^4, i.e. $\partial/\partial y^4$ is a Killing vector, if $\underline{h}_4 = 1$. Nevertheless, the decoupling property can be proven for the same assumptions but arbitrary $\underline{h}_4(x^k, y^4)$ with nontrivial dependence on y^4. We call this class of metrics to be with effective Killing symmetry because they result in systems of PDE (20) as for the Killing case but there are differences in (23) if $\underline{h}_4 \neq 1$. In order to simplify formulas, there will be used brief denotations for partial derivatives,

$$a^{\bullet} = \partial_1 = \partial a/\partial x^1, a' = \partial_2 = \partial a/\partial x^2, a^* = \partial_3 = \partial a/\partial y^3, a^{\circ} = \partial_4 = \partial a/\partial y^4.$$

Theorem 6. *The effective Einstein eqs (20) and nonholonomic constraints(23) for a metric* $\mathbf{g}$ *(10) with* $\omega = \underline{h}_3 = 1$ *and* $\underline{w}_i = \underline{n}_i = 0$ *and* $\underline{\mathbf{\Upsilon}} = 0$ *in matter source* $\mathbf{\Upsilon}^{\beta}{}_{\delta}$ *(21) are equivalent, respectively, to*

$$\widehat{R}^1_1 = \widehat{R}^2_2 = \frac{-1}{2g_1 g_2}[g_2^{\bullet\bullet} - \frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1'' - \frac{g_1' g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}] = -\ {}^v\Upsilon, \quad (25)$$

$$\widehat{R}^3_3 = \widehat{R}^4_4 = -\frac{1}{2h_3 h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] = -\Upsilon, \quad (26)$$

$$\widehat{R}_{3k} = \frac{w_k}{2h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] + \frac{h_4^*}{4h_4}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}\right) - \frac{\partial_k h_4^*}{2h_4} = 0, \quad (27)$$

$$\widehat{R}_{4k} = \frac{h_4}{2h_3} n_k^{**} + \left(\frac{h_4}{h_3} h_3^* - \frac{3}{2} h_4^*\right)\frac{n_k^*}{2h_3} = 0, \quad (28)$$

and

$$w_i^* = (\partial_i - w_i \partial_3)\ln|h_3|, (\partial_i - w_i \partial_3)\ln\sqrt{|h_4|} = 0, \partial_k w_i = \partial_i w_k, \quad (29)$$
$$n_k \underline{h}_4^{\circ} = \partial_k \underline{h}_4, n_i^* = 0, \partial_i n_k = \partial_k n_i.$$

Proof. See a sketch of proof in Appendix Appendix B; more details and computations for nonholonomic configurations in higher dimensions and the Einstein gravity are provided in Refs. [12,14,15]. □ □

The system of PDE (25)–(28) possess a very important decoupling property which is characteristic for various classes of modified gravity theories. Let us explain in brief this property. The equation (25) is for a 2–d metric which is always conformally flat and can be diagonalized. Choosing any value of a function g_1 for a prescribed source ${}^v\Upsilon$, we can find g_2, or inversely. The equation (26) contains only the first and second derivatives on $\partial/\partial y^3$ and relates two functions h_3 and h_4. The equations (27) consist a linear algebraic system for w_k if the coefficients h_a have been already defined as a solution of (26). Nevertheless, we have to consider additional constraints on w_i and h_4 solving a system of first order PDE on x^k and y^3 in order to find w_k resulting in zero torsion conditions (29). There are additional conditions on n_k. We shall analyze how the Levi–Civita conditions can be solved in very general forms in section 3.

Corollary 7. *The effective Einstein eqs (20) and (23) for a metric* $\mathbf{g}$ *(10) with* $\omega = h_4 = 1$ *and* $w_i = n_i = 0$ *and* $\mathbf{\Upsilon} = 0$ *in matter source* $\mathbf{\Upsilon}^{\beta}{}_{\delta}$ *(21), when such values do not depend on coordinate* y^3 *and posses one Killing symmetry on* $\partial/\partial y_3$*, are equivalent, respectively, to*

$$-\widehat{R}^1_1 = -\widehat{R}^2_2 = g_2^{\bullet\bullet} - \frac{g_1^{\bullet}g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1'' - \frac{g_1'g_2'}{2g_2} - \frac{(g_1')^2}{2g_1} = 2g_1g_2\ {}^v\Upsilon, \qquad (30)$$

$$\widehat{R}^3_3 = \widehat{R}^4_4 = -\frac{1}{2\underline{h}_3\underline{h}_4}[\underline{h}_3^{\circ\circ} - \frac{(\underline{h}_3^{\circ})^2}{2\underline{h}_3} - \frac{\underline{h}_3^{\circ}\underline{h}_4^{\circ}}{2\underline{h}_4}] = -\underline{\Upsilon}, \qquad (31)$$

$$\widehat{R}_{3k} = +\frac{\underline{h}_3}{2\underline{h}_4}\underline{w}_k^{\circ\circ} + \left(\frac{\underline{h}_3}{\underline{h}_4}\underline{h}_4^{\circ} - \frac{3}{2}\underline{h}_3^{\circ}\right)\frac{\underline{h}_k^{\circ}}{2\underline{h}_4} = 0, \qquad (32)$$

$$\widehat{R}_{4k} = \frac{\underline{n}_k}{2\underline{h}_3}[\underline{h}_3^{\circ\circ} - \frac{(\underline{h}_3^{\circ})^2}{2\underline{h}_3} - \frac{\underline{h}_3^{\circ}\underline{h}_4^{\circ}}{2\underline{h}_4}] + \frac{\underline{h}_3^{\circ}}{4\underline{h}_3}\left(\frac{\partial_k\underline{h}_3}{\underline{h}_3} + \frac{\partial_k\underline{h}_4}{\underline{h}_4}\right) - \frac{\partial_k h_3^{\circ}}{2h_3} = 0, \qquad (33)$$

$$\text{and } \underline{n}_i^{\circ} = (\partial_i - \underline{n}_i\partial_4)\ln|\underline{h}_4|, (\partial_i - \underline{n}_i\partial_4)\ln|\underline{h}_3| = 0,, \qquad (34)$$
$$(\partial_k - \underline{n}_k\partial_4)\underline{n}_i = (\partial_i - \underline{n}_i\partial_4)\underline{n}_k, \underline{w}_k h_3^{*} = \partial_k h_3, \underline{w}_i^{\circ} = 0, \partial_i\underline{w}_k = \partial_k\underline{w}_i.$$

Proof. It is similar to that for Theorem 6 but for $y^3 \to y^4$. We do not repeat such computations. □ □

The nonlinear systems of PDE corresponding to modified gravitational field equations (20) and (23) for metrics $\mathbf{g}$ (10) with Killing symmetry on $\partial/\partial y_4$, when $\omega = \underline{h}_3 = 1$ and $\underline{w}_i = \underline{n}_i = 0$ and $\underline{\mathbf{\Upsilon}} = 0$ in matter source $\mathbf{\Upsilon}^{\beta}{}_{\delta}$ (21), can be transformed into respective systems of PDE for data with Killing symmetry on $\partial/\partial y_3$, when $\omega = h_4 = 1$ and $w_i = n_i = 0$ and $\mathbf{\Upsilon} = 0$, if $h_3(x^i, y^3) \to \underline{h}_4(x^i, y^4)$, $h_4(x^i, y^3) \to \underline{h}_3(x^i, y^4)$, $w_k(x^i, y^3) \to \underline{n}_k(x^i, y^4)$ and $n_k(x^i, y^3) \to \underline{w}_k(x^i, y^4)$.

There is a possibility to preserve a N–adapted decoupling under "vertical" conform transforms.

Lemma 8. *The modified gravitational equations (26)–(28), do not change under a "vertical" conformal transform with nontrivial* $\omega(x^k, y^a)$ *to a d–metric (10) if there are satisfied the conditions*

$$\partial_k\omega - w_i\omega^{*} - n_i\omega^{\circ} = 0 \text{ and } \widehat{T}^a_{kb} = 0. \qquad (35)$$

Proof. We do not repeat here such details provided in Refs. [13,14] for $\underline{h}_4 = 1$ because a nontrivial $\underline{h}_4$ does not modify substantially the proof. The computations from Appendix Appendix B should be performed for coefficients $g_i(x^k), g_3 = h_3, g_4 = h_4\underline{h}_4, N_i^3 = w_i,\ N_i^4 = n_i$ are generalized to a nontrivial $\omega(x^k, y^a)$ with ${}^{\omega}g_3 = \omega^2 h_3$ and ${}^{\omega}g_4 = \omega^2 h_4 \underline{h}_4$. Using formulas (13), (A.1), (A.4) and (A.5), we get certain distortion relations for the Ricci d–tensors (A.9), ${}^{\omega}\widehat{R}^a{}_b = \widehat{R}^a{}_b + {}^{\omega}\widehat{Z}^a{}_b$ and ${}^{\omega}\widehat{R}_{bi} = \widehat{R}_{bi} = 0$, where $\widehat{R}^a{}_b$ and $\widehat{R}_{bi}$ are those computed for $\omega = 1$, i.e. (26)–(28). The values ${}^{\omega}\widehat{R}^a{}_b$ and ${}^{\omega}\widehat{Z}^a{}_b$ are defined by a nontrivial ω and computed using the same formulas. We get that ${}^{\omega}\widehat{Z}^a{}_b = 0$ if the conditions (35) are satisfied. □ □

The conditions of the Theorem 6, Corollary 7 and Lemma 8 result in a prove that

Theorem 9. *Any d–metric*

$$\mathbf{g} = g_i(x^k)dx^i \otimes dx^i + \omega^2(x^k, y^a)\left(h_3 \mathbf{e}^3 \otimes \mathbf{e}^3 + h_4\ \underline{h}_4 \mathbf{e}^4 \otimes \mathbf{e}^4\right), \\ \mathbf{e}^3 = dy^3 + w_i(x^k, y^3)dx^i,\ \mathbf{e}^4 = dy^4 + n_i(x^k)dx^i, \quad (36)$$

satisfying the PDE (25)– (29) and $\partial_k\omega - w_i\omega^* - n_i\omega^\circ = 0$*, or any d–metric*

$$\mathbf{g} = g_i(x^k)dx^i \otimes dx^i + \omega^2(x^k, y^a)\left(h_3\underline{h}_3 \mathbf{e}^3 \otimes \mathbf{e}^3 + \underline{h}_4 \mathbf{e}^4 \otimes \mathbf{e}^4\right), \\ \mathbf{e}^3 = dy^3 + \underline{w}_i(x^k)dx^i,\ \mathbf{e}^4 = dy^4 + \underline{n}_i(x^k, y^4)dx^i, \quad (37)$$

satisfying the PDE (30)– (34) and $\partial_k\omega - \underline{w}_i\omega^* - \underline{n}_i\omega^\circ = 0$*, define (in general, different) two classes of generic off–diagonal solutions of modified gravitational equations (20) and (23) with respective sources of type (21).*

Both ansatz of type (36) and (37) consist particular cases of parameterizations of metrics in the form (9). Via frame/coordinate transform into a finite region of a point ${}^0u \in \mathbf{V}$ any (pseudo) Riemannian metric can be represented in an above mentioned d–metric form. On Lorentz manifolds, only one of coordinates y^a is timelike, i.e. the solutions of type (36) and (37) can not be transformed mutually via nonholonomic frame deformations preserving causality.

2.3.2. *Effective linearization of Ricci tensors*

We can consider such local coordinates on an open region $U \subset \mathbf{V}$ when computing the N–adapted coefficients of the Riemann and Ricci d–tensors, see formulas (A.4) and (A.5), we can neglect contributions from quadratic terms of type $\widehat{\Gamma}\cdot\widehat{\Gamma}$ (preserving values of type $\partial_\mu\widehat{\Gamma}$). These are N–adapted analogs of normal coordinates when $\widehat{\Gamma}(u_0) = 0$ for points u_0, for instance, belonging to a line on U. Such conditions can be satisfied for decompositions of metrics and connections on a small parameter like it is explained in details in Ref. [12] (we shall consider decompositions on a small eccentricity parameter ε, for ellipsoid configurations, in Section 5). Other possibilities can be found if we impose nonholonomic constraints, for instance, of type $h_4^* = 0$ but for nonzero $h_4(x^k, y^3)$ and/or $h_4^{**}(x^k, y^3)$. Such constraints can be solved in non–explicit form and define a corresponding subclass of N–adapted frames. Considering additional nonholonomic deformations with a general decoupling with respect to a "convenient" system of reference/coordinates, we can deform the equations and solutions to configurations with contributions from $\widehat{\Gamma} \cdot \widehat{\Gamma}$ terms.

Theorem 10 (effective linearized decoupling). *The modified gravitational equations (20) and (23), via nonholonomic frame deformations to a metric* $\mathbf{g}$ *(10) and matter source* $\mathbf{\Upsilon}^{\beta}{}_{\delta}$ *(21), can be considered for an open region* $U \subset \mathbf{V}$ *when the contributions from terms of type* $\widehat{\Gamma} \cdot \widehat{\Gamma}$ *are small and we obtain and effective system of PDE with h–v–decoupling:*

$$\widehat{R}_1^1 = \widehat{R}_2^2 = \frac{-1}{2g_1 g_2}[g_2^{\bullet\bullet} - \frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1'' - \frac{g_1' g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}] = -\ {}^{v}\Upsilon, \tag{38}$$

$$\begin{aligned} \widehat{R}_3^3 = \widehat{R}_4^4 &= -\frac{1}{2h_3 h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] - \frac{1}{2\underline{h}_3 \underline{h}_4}[\underline{h}_3^{\circ\circ} - \frac{(\underline{h}_3^{\circ})^2}{2\underline{h}_3} - \frac{\underline{h}_3^{\circ} \underline{h}_4^{\circ}}{2\underline{h}_4}] \\ &= -\Upsilon - \underline{\Upsilon}, \end{aligned} \tag{39}$$

$$\begin{aligned} \widehat{R}_{3k} &= \frac{w_k}{2h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] + \frac{h_4^*}{4h_4}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}\right) - \frac{\partial_k h_4^*}{2h_4} \\ &\quad + \frac{\underline{h}_3}{2\underline{h}_4}\underline{n}_k^{\circ\circ} + \left(\frac{\underline{h}_3}{\underline{h}_4}\underline{h}_4^{\circ} - \frac{3}{2}\underline{h}_3^{\circ}\right)\frac{\underline{n}_k^{\circ}}{2\underline{h}_4} = 0, \end{aligned} \tag{40}$$

$$\begin{aligned} \widehat{R}_{4k} &= \frac{\underline{w}_k}{2\underline{h}_3}[\underline{h}_3^{\circ\circ} - \frac{(\underline{h}_3^{\circ})^2}{2\underline{h}_3} - \frac{\underline{h}_3^{\circ} \underline{h}_4^{\circ}}{2\underline{h}_4}] + \frac{\underline{h}_3^{\circ}}{4\underline{h}_3}\left(\frac{\partial_k \underline{h}_3}{\underline{h}_3} + \frac{\partial_k \underline{h}_4}{\underline{h}_4}\right) - \frac{\partial_k \underline{h}_3^{\circ}}{2\underline{h}_3} \\ &\quad + \frac{h_4}{2h_3}n_k^{**} + \left(\frac{h_4}{h_3}h_3^* - \frac{3}{2}h_4^*\right)\frac{n_k^*}{2h_3} = 0, \end{aligned} \tag{41}$$

$$\begin{aligned} w_i^* &= (\partial_i - w_i \partial_3)\ln|h_3|, (\partial_i - w_i\partial_3)\ln|h_4| = 0 \\ (\partial_k - w_k\partial_3)w_i &= (\partial_i - w_i\partial_3)w_k, n_i^* = 0, \partial_i n_k = \partial_k n_i, \\ \underline{w}_i^{\circ} &= 0,\ \partial_i \underline{w}_k = \partial_k \underline{w}_i, (\partial_k - \underline{n}_k\partial_4)\underline{n}_i = (\partial_i - \underline{n}_i\partial_4)\underline{n}_k, \\ \underline{n}_i^{\circ} &= (\partial_i - \underline{n}_i\partial_4)\ln|\underline{h}_4|, (\partial_i - \underline{n}_i\partial_4)\ln|\underline{h}_3| = 0 \end{aligned} \tag{42}$$

$$\mathbf{e}_k\omega = \partial_k\omega - (w_i + \underline{w}_i)\,\omega^* - (n_i + \underline{n}_i)\,\omega^{\circ} = 0. \tag{43}$$

Proof. It follows from the Theorems 6 and 9 for any superposition of ansatz (36) and (37) resulting into metrics of type (9). □ □

In general, the solutions defined by a system (38)–(43) can not be transformed into solutions parameterized by an ansatz (36) and/or (37). As we shall prove in Section 3, the general solutions of the such systems of PDE are determined by corresponding sets of generating and integration functions. A solution for (38)–(43) contains a larger set of $h - v$–generating functions than those with some N–coefficients stated to be zero.

2.4. *Decoupling of MGYMH equations*

We write for $\mathbf{T}_{\beta\delta}$ in (18) the corresponding values for the energy–momentum tensors of the Yang–Mills, YM, and Higgs, H, fields, when

$$\Upsilon_{\beta\delta} = 8\pi \ {}^{ef}\eta \ G \ \ {}^{m}\mathbf{T}_{\beta\delta} + \ {}^{ef}\mathbf{T}_{\beta\delta},$$

where we introduced the coefficient 8π in order to get in the Einstein limit solutions parameterized in the form [19,20,21,22]. A variational N–adapted procedure can be elaborated on a manifold $\mathbf{V}$ when the operator $\widehat{\mathbf{D}}$ is used instead of ∇ and all computations are performed with respect N–adapted bases (3) and (4). It is completely similar to that for the Levi–Civita connection but with N–elongated partial derivatives for a gravitating non–Abbelian SU(2) gauge field $\mathbf{A} = \mathbf{A}_\mu \mathbf{e}^\mu$ coupled to a triplet Higgs field Φ. We derive this system of modified gravitational and matter field equations (in brief, MGYMH):

$$\widehat{\mathbf{R}}_{\ \beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta} \ {}^{s}R = 8\pi \ {}^{ef}\eta G \left(\ {}^{H}T_{\beta\delta} + \ {}^{YM}T_{\beta\delta} \right) + \ {}^{ef}\mathbf{T}_{\beta\delta}, \quad (45)$$

$$D_\mu(\sqrt{|g|}F^{\mu\nu}) = \frac{1}{2} ie\sqrt{|g|}[\Phi, D^\nu \Phi], \quad (46)$$

$$D_\mu(\sqrt{|g|}\Phi) = \lambda\sqrt{|g|}(\ \Phi^2_{[0]} - \Phi^2)\Phi, \quad (47)$$

where the stress–energy tensors for the YM and H fields are computed

$${}^{YM}T_{\beta\delta} = 2Tr\left(\mathbf{g}^{\mu\nu}F_{\beta\mu}F_{\delta\nu} - \frac{1}{4}\mathbf{g}_{\beta\delta}F_{\mu\nu}F^{\mu\nu}\right), \quad (48)$$

$${}^{H}T_{\beta\delta} = Tr[\frac{1}{4} \ (D_\delta\Phi \ D_\beta\Phi + D_\beta\Phi \ D_\delta\Phi) - \frac{1}{4}\mathbf{g}_{\beta\delta}D_\alpha\Phi \ D^\alpha\Phi] - \mathbf{g}_{\beta\delta}\mathcal{V}(\Phi). \quad (49)$$

The value ${}^{ef}\mathbf{T}_{\beta\delta}$ in (45) is the same as in (19) but for zero pressure, $p = 0$. The nonholonomic and modified gravitational interactions of matter fields and the constants in (45)–(47) are treated as follows: The non–Abbelian gauge field with derivative $D_\mu = \mathbf{e}_\mu + ie[\mathbf{A}_\mu,]$ is changed into $\widehat{\mathcal{D}}_\delta = \widehat{\mathbf{D}}_\delta + ie[\mathbf{A}_\delta,]$. The vector field $\mathbf{A}_\delta$ is characterised by curvature

$$F_{\beta\mu} = \mathbf{e}_\beta \mathbf{A}_\mu - \mathbf{e}_\mu \mathbf{A}_\beta + ie[\mathbf{A}_\beta, \mathbf{A}_\mu], \quad (50)$$

where e is the coupling constant, $i^2 = -1$, and $[\cdot,\cdot]$ is used for the commutator. We also consider that the value $\Phi_{[0]}$ in (47) is the vacuum expectation of the Higgs field which determines the mass ${}^{H}M = \sqrt{\lambda}\eta$; the value λ is the constant of scalar field self–interaction with potential $\mathcal{V}(\Phi) = \frac{1}{4}\lambda Tr(\Phi^2_{[0]} - \Phi^2)^2$; the gravitational constant G defines the Plank mass $M_{Pl} = 1/\sqrt{G}$ and it is also the mass of a gauge boson, ${}^{W}M = ev$.

Let us consider that a "prime" solution is known for the system (45)–(47) (given by data for a diagonal d–metric $^\circ\mathbf{g} = [\ ^\circ g_i(x^1), \ ^\circ h_a(x^k), \ ^\circ N_i^a = 0]$ and matter fields $^\circ A_\mu(x^1)$ and $^\circ\Phi(x^1)$, for instance, of type constructed in Ref. [23] (see also Appendix B.4)). We suppose that there are satisfied the following conditions:

(1) The d–metric ${}^{\eta}\mathbf{g}$ with nontrivial N–coefficients for ${}^{\circ}\mathbf{g} \rightarrow {}^{\eta}\mathbf{g}$ is parameterized by an ansats of type (9),

$$\begin{aligned}\mathbf{g} &= \eta_i(x^k)\ {}^{\circ}g_i(x^1)dx^i \otimes dx^i + \eta_a(x^k, y^a)\ {}^{\circ}h_a(x^1, x^2)\mathbf{e}^a \otimes \mathbf{e}^a \\ &= g_i(x^k)dx^i \otimes dx^i + \omega^2(x^k, y^b)h_a(x^k, y^a)\mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dy^3 + [w_i + \underline{w}_i]dx^i,\ \mathbf{e}^4 = dy^3 + [n_i + \underline{n}_i]dx^i.\end{aligned} \tag{51}$$

(2) The non–Abbelian gauge fields are nonholonomically deformed as

$$A_\mu(x^i, y^3) = {}^{\circ}A_\mu(x^1) + {}^{\eta}A_\mu(x^i, y^a), \tag{52}$$

where ${}^{\circ}A_\mu(x^1)$ is defined by an ansatz (B.17) and ${}^{\eta}A_\mu(x^i, y^a)$ are any functions

$$F_{\beta\mu} = {}^{\circ}F_{\beta\mu}(x^1) + {}^{\eta}F_{\beta\mu}(x^i, y^a) = s\sqrt{|g|}\varepsilon_{\beta\mu}, \tag{53}$$

for $s = const$ and $\varepsilon_{\beta\mu}$ being the absolute antisymmetric tensor. The gauge field curvatures $F_{\beta\mu}$, ${}^{\circ}F_{\beta\mu}$ and ${}^{\eta}F_{\beta\mu}$ are computed by introducing (B.17) and (52) into (50). It should be emphasized that an antisymmetric tensor $F_{\beta\mu}$ (53) solves the equations $D_\mu(\sqrt{|g|}F^{\mu\nu}) = 0$; we can always determine ${}^{\eta}F_{\beta\mu}$ and ${}^{\eta}A_\mu$, for any given ${}^{\circ}A_\mu$ and ${}^{\circ}F_{\beta\mu}$.

(3) The scalar field is nonholonomically modified by gravitational and gauge field interactions ${}^{\circ}\Phi(x^1) \rightarrow \Phi(x^i, y^a) = {}^{\Phi}\eta(x^i, y^a)\ {}^{\circ}\Phi(x^1)$ by a polarization ${}^{\Phi}\eta$ is such way that

$$D_\mu\Phi = 0 \text{ and } \Phi(x^i, y^a) = \pm\Phi_{[0]}. \tag{54}$$

Such nonholonomic modifications of the nonlinear scalar field is not trivial even with respect to N–adapted frames $\mathcal{V}(\Phi) = 0$ and ${}^{H}T_{\beta\delta} = 0$, see formula (48). For ansatz (51), the equations (54) transform into

$$\begin{aligned}(\partial/\partial x^i - A_i)\Phi &= (w_i + \underline{w}_i)\Phi^* + (n_i + \underline{n}_i)\Phi^\circ, \\ \left(\partial/\partial y^3 - A_3\right)\Phi &= 0,\ \left(\partial/\partial y^4 - A_4\right)\Phi = 0.\end{aligned} \tag{55}$$

So, a nonolonomically constrained/deformed Higgs Φ field (depending in non–explicit form on two variables because of constraint (54)) modifies indirectly the off–diagonal components of the metric via $w_i + \underline{w}_i$ and $n_i + \underline{n}_i$ and conditions (55) for ${}^{\eta}A_\mu$. Such modifications can compensate f–modifications.

(4) The non–Abbelian gauge fields (53) with the potential A_μ (52) modified nonholonomically by Φ subjected to the conditions (54) and with gravitational f–modifications determine exact solutions of the system (39) and (40) if the metric ansatz is chosen to be in the form (51). The energy–momentum tensor is computed[f] ${}^{YM}T^\alpha_\beta = -4s^2\delta^\alpha_\beta m$ (see similar results in sections 3.2 and 6.51 in Ref. [24]). Such (modified) gravitationally interacting gauge and Higgs fields, with respect to N–adapted frames, result in an effective cosmological constant ${}^{s}\lambda = 8\pi s^2$ which should be added to a respective source (21).

[f] such a calculus in coordinate frames is provided in sections 3.2 and 6.51 in Ref. [24]

We conclude that an ansatz $\mathbf{g} = [\eta_i\ ^{\circ}g_i, \eta_a\ ^{\circ}h_a; w_i, n_i]$ (51) and certain gauge–scalar configurations (A, Φ) subjected to above mentioned conditions 1-4 define a decoupling of the system (45)–(47) in a form stated respectively by the Theorems 6, 9, and/or 10 if the sources (21) are redefined in the form

$$\mathbf{\Upsilon}^{\beta}_{\ \delta} = diag[\mathbf{\Upsilon}_{\alpha}] \rightarrow \mathbf{\Upsilon}^{\beta}_{\ \delta} +\ ^{YM}T^{\beta}_{\ \delta} = diag[\mathbf{\Upsilon}_{\alpha} - 4s^2\delta^{\alpha}_{\beta}].$$

In N–adapted frames the contributions of effective f–sources and matter fields is defined by an effective cosmological constant $\ ^{s}\lambda$.

3. Off–Diagonal Solutions for Modified Gravitational YMH Eqs

In this section, we show how the decoupling property of the MGYMH equations allows us to integrate such PDE in very general forms depending on properties of coefficients of ansatz for metrics.

3.1. *Generating solutions with weak one Killing symmetry*

We prove that the MGYMH equations encoding gravitational and YMH interactions and satisfying the conditions of Theorem 6 can be integrated in general forms for $h_a^* \neq 0$ and certain special cases with zero and non–zero sources (21). In general, such generic off–diagonal metrics are determined by generating functions depending on three/four coordinates.

3.1.1. *(Non) vacuum metrics with $h_a^* \neq 0$*

For ansatz (9) with data $\omega = 1, \underline{h}_3 = 1, \underline{w}_i = 0$ and $\underline{n}_i = 0$ for (10), when $h_a^* \neq 0$, and the condition that the source

$$\mathbf{\Upsilon}^{\beta}_{\ \delta} = diag[\mathbf{\Upsilon}_{\alpha} : \mathbf{\Upsilon}^1_{\ 1} = \mathbf{\Upsilon}^2_{\ 2} = \Upsilon(x^k, y^3) - 4s^2; \mathbf{\Upsilon}^3_{\ 3} = \mathbf{\Upsilon}^4_{\ 4} =\ ^{v}\Upsilon(x^k) - 4s^2], \quad (1)$$

is not zero, the solutions of Einstein eqs can be constructed following

Theorem 11. *The MGYMH equations (25)– (28) with source (1) can be integrated in general forms by metrics*

$$\mathbf{g} = \epsilon_i e^{\psi(x^k)} dx^i \otimes dx^i + \frac{|\tilde{\Theta}^*|^2}{\breve{\Upsilon}\tilde{\Theta}^2}\mathbf{e}^3 \otimes \mathbf{e}^3 - \frac{\tilde{\Theta}^2}{|\Lambda|}\underline{h}_4(x^k, y^4)\mathbf{e}^4 \otimes \mathbf{e}^4,$$
$$\mathbf{e}^3 = dy^3 + \partial_i \widetilde{K}(x^k, y^3)dx^i,\ \mathbf{e}^4 = dy^4 + \partial_i n(x^k)dx^i, \quad (2)$$

with coefficients determined by generating functions $\psi(x^k), \tilde{\Theta}(x^k, y^3),\ \tilde{\Theta}^* \neq 0$, $n_i(x^k)$ and $\underline{h}_4(x^k, y^4)$, and effective cosmological constant Λ and source $\Upsilon - 4s = \breve{\Upsilon}(x^k) \neq 0$ following recurrent formulas and conditions

$$\epsilon_1 \psi^{\bullet\bullet} + \epsilon_2 \psi'' = 2\ [\ ^{v}\Upsilon - 4s^2]; \quad (3)$$
$$h_4 = -K^2 = -\left[\ _0K^2 + \int dy^3 \frac{(\Theta^2)^*}{4(\Upsilon - 4s^2)}\right] \quad (4)$$

$$
\begin{aligned}
&= ({}_0K^2 + \tilde{\Theta}^2/\Lambda);\\
h_3 &= B^2 = 4(K^*)^2/\Theta^2\\
&= (\Theta^*)^2/4(\Upsilon - 4s^2)^2 \left[{}_0K^2 + \int dy^3 (\Theta^2)^*/4(\Upsilon - 4s^2)\right]\\
&= |\tilde{\Theta}^*|^2/\int dy^3(\Upsilon - 4s^2)(\tilde{\Theta}^2)^* = |\tilde{\Theta}^*|^2/\check{\Upsilon}\tilde{\Theta}^2; \quad (5)\\
w_i &= \partial_i \phi/\phi^* = \partial_i\Theta/\Theta^*, \quad (6)\\
&= \partial_i K/K^* = \partial_i|\tilde{\Theta}|/|\tilde{\Theta}|^*, \ \textit{if}\ \Upsilon - 4s = \check{\Upsilon}(x^k).
\end{aligned}
$$

where the constraints

$$
\begin{aligned}
w_i^* &= (\partial_i - w_i\partial_3)\ln|h_3|, (\partial_i - w_i\partial_3)\ln|h_4| = 0, \quad (7)\\
\partial_k w_i &= \partial_i w_k, n_k \underline{h}_4^\circ = \partial_k \underline{h}_4, \partial_i n_k = \partial_k n_i.
\end{aligned}
$$

are used for the Levi–Civita configurations.

Proof. We sketch a proof which related to similar ones in [13,14,12] if $\underline{h}_4 = 1$. The N–adapted coefficients of a metric (7) are parameterized in the form

$$
g_i = e^{\psi(x^k)}, g_a = \omega(x^k, y^b)h_a(x^k, y^3),\ N_i^3 = w_i(x^k, y^3), N_i^4 = n_i(x^k), \quad (8)
$$

considering that for certain frame/coordinate transforms we can satisfy the conditions $h_a^* \neq 0, \Upsilon_{2,4} \neq 0$.

- We introduce the functions

$$
\phi(x^k, y^3) := \ln\left|\frac{h_4^*}{\sqrt{|h_3 h_4|}}\right|,\ \Theta := e^\phi \quad (9)
$$

$$
\gamma := \left(\ln\frac{|h_4|^{3/2}}{|h_3|}\right)^*,\quad \alpha_i = h_4^*\partial_i\phi,\ \beta = h_4^*\phi^* \quad (10)
$$

The system of equations (25)–(28) transforms into

$$
\begin{aligned}
\psi^{\bullet\bullet} + \psi'' &= 2[\ {}^v\Upsilon - 4s^2], \quad (11)\\
\phi^* h_4^* &= 2h_3h_4\left[\Upsilon - 4s^2\right] \quad (12)\\
\beta w_i - \alpha_i &= 0, \quad (13)\\
n_i^{**} + \gamma n_i^* &= 0, \quad (14)\\
\partial_i\omega - (\partial_i\phi/\phi^*)\omega^* - n_i\omega^\diamond &= 0, \quad (15)
\end{aligned}
$$

- A horizontal metric $g_i(x^2)$ is for a 2–d subspace and can be represented in a conformally flat form $\epsilon_i e^{\psi(x^k)}dx^i \otimes dx^i$. For such a h–metric, the equation (11) is a 2-d Laplace equation which can be solved exactly if a source ${}^v\Upsilon(x^k) - 4s^2$ is prescribed from formulas (21).

- For un–known two functions $K=+\sqrt{|h_4|}, B=+\sqrt{|h_3|}$, the system of equations (9) and (12) can be written in the form

$$\Theta^* K^* = (\Upsilon - 4s^2)B^2\Theta K, \quad (16)$$
$$\Theta B = 2K^*. \quad (17)$$

We introduce ΘB from (17) in the right of (16) for $A^* \neq 0$, and find $2AB\Upsilon = \Phi^*$ and devide to (17) with nonzero coefficients. We express

$$(K^2)^* = (\Theta^2)^*/4(\Upsilon - 4s^2). \quad (18)$$

Integrating on y^3, we find (4) for an integration function ${}_0K(x^k)$, which can be included in $\tilde{\Theta}$, and $\epsilon_4 = \pm 1$ depending on signature of the metric. We introduced in that formula an effective cosmological constant Λ and re–defined the generating function, $\Theta \rightarrow \tilde{\Theta}$,

$$\frac{(\Theta^2)^*}{4(\Upsilon - 4s^2)} = \frac{(\tilde{\Theta}^2)^*}{\Lambda}, \quad (19)$$

choosing $K = \tilde{K} = |\tilde{\Theta}|/\sqrt{|\Lambda|}$. Using (17), (4) and (18), rewritten in the form $K^*K = \Theta^*\Theta/4(\Upsilon - 4s^2)$, we obtain $B = 2K^*/\Theta$, i.e. (5). That formula is written for $\Upsilon - 4s^2 = \check{\Upsilon}(x^k)$ when we can transform $h_a[\ {}_0K, \Theta, \Upsilon - 4s^2] \rightarrow h_a[\tilde{\Theta}, \Lambda]$, for off–diagonal configurations determined by a cosmological constant Λ and generating function $\tilde{\Theta}(x^k, y^3)$.

- The algebraic equations for w_i can be solved by introducing the coefficients (10) in (13) for the generating function ϕ, or using equivalent variables,

$$w_i = \partial_i \phi/\phi^* = \partial_i \Theta/\Theta^* \quad (20)$$
$$= \partial_i K/K^* = \partial_i |\tilde{\Theta}|/|\tilde{\Theta}|^*, \text{ if } \Upsilon - 4s^2 = \check{\Upsilon}(x^k). \quad (21)$$

- Integrating two times on y^3, we obtain the solution of (14):

$$n_k = {}_1n_k + {}_2n_k \int dy^3\ h_3/(\sqrt{|h_4|})^3 = {}_1n_k + {}_2n_k \int dy^3 K^2/B^3,$$

for integration functions ${}_1n_k(x^i),\ {}_2n_k(x^i)$.

- The nonholonomic Levi–Civita conditions (7) can not be solved in explicit form for arbitrary data $(K, \Upsilon - 4s^2)$, or $(\Theta, \Upsilon - 4s^2)$, and arbitrary integration functions ${}_1n_k$ and ${}_2n_k$; we can fix ${}_2n_k = 0$ and ${}_1n_k = \partial_k n$ with a function $n = n(x^k)$. We emphasize that $(\partial_i - w_i\partial_3)\Theta \equiv 0$ for any $\Theta(x^k, y^3)$ if w_i is computed following formula (20). Introducing a new functional $H(\Theta)$ instead of Θ, we obtain $(\partial_i - w_i\partial_3)H = \frac{\partial H}{\partial \Theta}(\partial_i - w_i\partial_3)\Theta = 0$. Any formula (4) for functionals of type $h_4 = H(|\tilde{\Theta}(\Theta)|)$, we solve always the equations $(\partial_i - w_i\partial_3)h_4 = 0$, which is equivalent to the second system of equations in (7) because $(\partial_i - w_i\partial_3)\ln\sqrt{|h_4|} \sim (\partial_i - w_i\partial_3)h_4$. We compute for the left part of the second equation, $(\partial_i - w_i\partial_3)\ln\sqrt{|h_4|} = 0$, for a subclass of generating functions $\Theta = \check{\Theta}$ for which

$$(\partial_i \check{\Theta})^* = \partial_i \check{\Theta}^* \quad (23)$$

and (4). The first system of equations in (7) are solved in explicit form if w_i are determined by formulas (21), and $h_3[\tilde{\Theta}]$ and $h_4[\tilde{\Theta},\tilde{\Theta}^*]$ are respectively for (4) and (5) when $\Upsilon - 4s = \check{\Upsilon}(x^k)$. We can write write the formulas

$$w_i = \partial_i|\tilde{\Theta}|/|\tilde{\Theta}|^* = \partial_i|\ln\sqrt{|h_3|}|/|\ln\sqrt{|h_3|}|^*$$

if $\tilde{\Theta} = \tilde{\Theta}(\ln\sqrt{|h_3|})$ and $h_3[\tilde{\Theta}[\check{\Theta}]]$. Taking derivative ∂_3 on both sides of previous equation, we compute

$$w_i^* = \frac{(\partial_i|\ln\sqrt{|h_3|}|)^*}{|\ln\sqrt{|h_3|}|^*} - w_i\frac{|\ln\sqrt{|h_3|}|^{**}}{|\ln\sqrt{|h_3|}|^*}.$$

This way we are able to construct generic off–diagonal configurations with $w_i^* = (\partial_i - w_i\partial_3)\ln\sqrt{|h_3|}$, which is necessary for zero torsion conditions, if the constraints (23) are imposed. The conditions $\partial_k w_i = \partial_i w_k$ from the second line in (7) are satisfied by any

$$\check{w}_i = \partial_i\check{\Theta}/\check{\Theta}^* = \partial_i\widetilde{K}, \quad (26)$$

when a nontrivial $\widetilde{K}(x^k,y^3)$ exists. □ □

The solutions constructed in Theorem 11, and those derived following Corollary 7 are very general ones and contain as particular cases all known exact solutions for (non) holonomic Einstein spaces with Killing symmetries and f–modifications studied in this paper. They can be generalized to include arbitrary finite sets of parameters, see [12].

For arbitrary K and $\Upsilon - 4s$, and related Θ, or $\tilde{\Theta}$, and Λ, we can generate off–diagonal solutions of (25)–(28) with nonholonomically induced torsion completely determined by the metric structure,

$$\begin{aligned} ds^2 &= e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + K^2[dy^3 + \frac{\partial_i\Theta}{\Theta^*}dx^i]^2 \\ &-B^2[dt + ({}_1n_k + {}_2n_k\int dy^3 K^2/B^3)dx^k]^2, \end{aligned} \quad (27)$$

where the generating functions K, B and Θ are related via formulas (4) and (5) but not subjected to the conditions (7).

3.1.2. *Off–diagonal effective vacuum EYMH configurations*

We can consider a subclass of generic off–diagonal MGYMH interactions which can be encoded as effective vacauum Einstein manifolds when $\Upsilon = 4s^2$. In general, such classes of solutions depend parametrically on $\Upsilon - 4s^2$ and do not have a smooth limit from non-vacuum to vacuum models.

Corollary 12. *The effective vacuum solutions for the EYHM systems with ansatz for metrics of type (2) with vanishing source (1) are parameterized in the form*

$$\begin{aligned} &\mathbf{g} = \epsilon_i e^{\psi(x^k)}dx^i\otimes dx^i + h_3(x^k,y^3)\mathbf{e}^3\otimes\mathbf{e}^3 + h_4(x^k,y^3)\underline{h}_4(x^k,y^4)\mathbf{e}^4\otimes\mathbf{e}^4, \\ &\mathbf{e}^3 = dy^3 + w_i(x^k,y^3)dx^i,\ \mathbf{e}^4 = dy^4 + n_i(x^k)dx^i, \end{aligned} \quad (28)$$

where coefficients are defined by solutions of the system

$$\ddot{\psi} + \psi'' = 0, \quad (29)$$

$$\phi^* \ h_4^* = 0, \quad (30)$$

$$\beta w_i + \alpha_i = 0, \quad (31)$$

where the coefficients are subjected additionally to the zero–torsion conditions (7).

Proof. The equations (29)– (31) are, respectively, (11)– (13) with zero sources. To solve (29) we can take $\psi = 0$ or consider a trivial 2-d wave equation if one of coordinates x^k is timelike.

There are two classes of solutions for (30):

The first one is to consider that $h_4 = h_4(x^k)$, i.e. $h_4^* = 0$, which states that the equation (30) has solutions for arbitrary function $h_3(x^k, y^3)$ and arbitrary N–coefficients $w_i(x^k, y^3)$, see (10). The functions h_3 and w_i can be taken as generation ones which should be constrained only by the conditions (7). The second equations in such conditions constrain substantially the class of admissible w_i if h_4 depends only on x^k. Nevertheless, h_3 can be an arbitrary one generating solutions which can be extended for nontrivial sources $\underline{\Upsilon}$ and systems (30)– (34) and/or (38)– (43).

The second class of solutions can be generated after corresponding coordinate transforms, $\phi = \ln \left| h_4^* / \sqrt{|h_3 h_4|} \right| = \ {}^0\phi = const, \ \phi^* = 0$ and $h_4^* \neq 0$. We can solve (30) if

$$\sqrt{|h_3|} = \ {}^0h(\sqrt{|h_4|})^*, \quad (32)$$

for ${}^0h = const \neq 0$. Such v–metrics are generated by any function $\varpi(x^i, y^3)$, with $\varpi^* \neq 0$, when

$$h_4 = -\varpi^2 \left(x^i, y^3\right) \text{ and } h_3 = (\ {}^0h)^2 \ \left[\varpi^* \left(x^i, y^3\right)\right]^2 ; \quad (33)$$

for $N_i^a \to 0$ we obtain diagonal metrics with signature $(+,+,+,-)$. The coefficients $\alpha_i = \beta = 0$ in (31) and $w_i(x^k, y^3)$ can be any functions subjected to the conditions (7), or equivalently to

$$w_i^* = 2\partial_i \ln|\varpi| - 2w_i(\ln|\varpi|)^*, \quad (34)$$
$$\partial_k w_i - \partial_i w_k = 2(w_k \partial_i - w_i \partial_k)\ln|\varpi|,$$

for any $n_i(x^k)$ when $\partial_i n_k = \partial_k n_i$. Constraints of type $n_k \underline{h}_4^\circ = \partial_k \underline{h}_4$ (B.9) have to be imposed for a nontrivial multiple $\underline{h}_4$. □ □

Using Corollary 7, the "dual" ansatz to (28) with $y^3 \to y^4$ and $y^4 \to y^3$ can be used to generate effective vacuum solutions with weak Killing symmetry on $\partial/\partial y^3$.

3.2. *Effective EYMH configurations with non–Killing symmetries*

The Theorem 9 can be applied for constructing non–vacuum and effective vacuum solutions of the EYMH equations depending on all coordinates without explicit Killing symmetries.

3.2.1. *Non–vacuum off–diagonal solutions*

We can generate such YMH Einstein manifolds following

Corollary 13. *An ansatz of type (36) with a d–metric*

$$\mathbf{g} = \epsilon_i e^{\psi(x^k)} dx^i \otimes dx^i + \omega^2 [\frac{|\tilde{\Theta}^*|^2}{\breve{\Upsilon}\tilde{\Theta}^2} \mathbf{e}^3 \otimes \mathbf{e}^3 - \frac{\tilde{\Theta}^2}{|\Lambda|} \underline{h}_4(x^k, y^4) \mathbf{e}^4 \otimes \mathbf{e}^4],$$
$$\mathbf{e}^3 = dy^3 + \partial_i \widetilde{K}(x^k, y^3) dx^i, \ \mathbf{e}^4 = dy^4 + \partial_i n(x^k) dx^i,$$

for ${}^v\Upsilon = \Upsilon = {}^0\Upsilon = const$*, where the coefficients are subjected to conditions (3)–(7) and*

$$\partial_k \omega + (\partial_i \phi / \phi^*) \omega^* - n_i \omega^\circ = 0,$$

defines solutions of the Einstein equations $R_{\alpha\beta} = ({}^0\Upsilon - 4s^2) g_{\alpha\beta}$ *with nonholonomic interactions anf f–modifications of YMH fields encoded effectively into the vacuum structure of GR with nontrivial cosmological constant,* ${}^0\Upsilon - 4s^2 \neq 0$.

Proof. We have to consider ${}^v\Upsilon = \Upsilon = {}^0\Upsilon = const$ in the Theorems 9 and Corollary 11. □ □

Solutions of type (37) can be generated for conformal factors being solutions of

$$\partial_k \omega - \underline{w}_i(x^k) \omega^* + (\partial_i \underline{\phi} / \underline{\phi}^\circ) \omega^\circ = 0$$

with respective "dual" generating functions ω and $\underline{\phi}$ when the data (3)–(7) are re–defined for solutions with weak Killing symmetry on $\partial/\partial y^3$.

3.2.2. *Effective vacuum off–diagonal solutions for f-modifications*

Vacuum Einstein spaces encoding nonholonomic interactions of MGYMH fields can be constructed using

Corollary 14. *An ansatz of type (36) with d–metric*

$$\mathbf{g} = \epsilon_i e^{\psi(x^k)} dx^i \otimes dx^i + \omega^2(x^k, y^a) [({}^0h)^2 \ \left[\varpi^* \left(x^i, y^3\right)\right]^2 \mathbf{e}^3 \otimes \mathbf{e}^3$$
$$- \varpi^2 \left(x^i, y^3\right) \underline{h}_4(x^k, y^4) \mathbf{e}^4 \otimes \mathbf{e}^4],$$
$$\mathbf{e}^3 = dy^3 + w_i(x^k, y^3) dx^i, \ \mathbf{e}^4 = dy^4 + n_i(x^k) dx^i,$$

where the coefficients are subjected to conditions (32)–(34), (7) and

$$\partial_k \omega - w_i \omega^* - n_i \omega^\circ = 0,$$

define generic off–diagonal solutions of $R_{\alpha\beta} = 0$.

Proof. It is a consequence of Theorem 9 and Corollary 12. □ □

Metric of class (37) are generated if the conformal factor is a solution of

$$\partial_k \omega - \underline{w}_i(x^k)\omega^* - \underline{n}_i(x^k,y^4)\omega^\circ = 0$$

with respective "dual" generating functions $\omega(x^k,y^a)$ and $\underline{\phi}(x^k,y^4)$; the data and conditions (32)–(34) and (7) are reconsidered for ansatz with weak Killing symmetry on $\partial/\partial y^3$.

4. f–modified and YMH Deformations of Black Holes

In modified gravity, possible gauge–Higgs nonholonomic interactions define off–diagonal deformations (for instance, of rotoid type) of Schwarzschild black holes. In this section, we study effective EYMH configurations when ${}^v\Upsilon = \Upsilon$ and $\Upsilon + {}^s\lambda = 0$. Nonholonomic deformations can be derived from any "prime" data $\left({}^\circ\mathbf{g},\ {}^\circ\mathbf{A}_\mu,\ {}^\circ\Phi\right)$ stating, for instance, a diagonal cosmological monopole and non–Abbelian black hole configuration in [23]. We can chose such a constant s for ${}^s\lambda$ when the effective source is zero (if ${}^s\lambda < 0$, this is possible for $\Upsilon > 0$). The resulting nonholonomic matter field configurations $\mathbf{A}_\mu = {}^\circ\mathbf{A}_\mu + {}^\eta\mathbf{A}_\mu$ (52), $F_{\mu\nu} = s\sqrt{|\mathbf{g}|}\varepsilon_{\mu\nu}$ (53) and $\Phi = {}^\Phi\eta\ {}^\circ\Phi$ subjected to the conditions (54) are encoded as vacuum off–diagonal polarizations into solutions of equations (38)–(39).

4.1. *(Non) holonomic and f–modified non–Abbelian effective vacuum spaces*

We have to construct off–diagonal solutions of the Einstein equations for the canonical d–connection taking the vacuum equations $\widehat{\mathbf{R}}_{\alpha\beta} = 0$ and ansatz $\mathbf{g}$ (51) with coefficients satisfying the conditions

$$\begin{aligned}
&\epsilon_1 \psi^{\bullet\bullet}(r,\theta) + \epsilon_2 \psi^{''}(r,\theta) = 0; &(1)\\
&h_3 = \pm e^{-2\ {}^0\phi}\frac{(h_4^*)^2}{h_4} \text{ for a given } h_4(r,\theta,\varphi),\ \phi(r,\theta,\varphi) = {}^0\phi = const;\\
&w_i = w_i(r,\theta,\varphi),\ \text{for any such functions if } \lambda = 0;\\
&n_i = \left\{ \begin{array}{lll} {}^1n_i(r,\theta) + {}^2n_i(r,\theta)\int (h_4^*)^2 |h_4|^{-5/2} dv, & \text{if} & n_i^* \neq 0;\\ {}^1n_i(r,\theta), & \text{if} & n_i^* = 0, \end{array}\right.
\end{aligned}$$

when h_4 and w_i are considered as generating functions. In general, such effective vacuum solutions can be not generated in limits $\Upsilon + {}^s\lambda \to 0$ because of singularity of coefficients, for instance, for a class of solutions (2) with coefficients (3)–(5).

Imposing additional constraints on coefficients of d–metric, for $e^{-2\ {}^0\phi} = 1$, as solutions of (7),

$$h_3 = \pm 4\left[\left(\sqrt{|h_4|}\right)^*\right]^2, \quad h_4^* \neq 0; \qquad (2)$$

$$w_1 w_2 \left(\ln|\frac{w_1}{w_2}|\right)^* = w_2^\bullet - w_1',\ w_i^* \neq 0;\ w_2^\bullet - w_1' = 0,\ w_i^* = 0;$$

$${}^1n_1'(r,\theta) - {}^1n_2^\bullet(r,\theta) = 0,\ n_i^* = 0, \qquad (3)$$

we generate effective vacuum solutions of the Einstein equations for the Levi–Civita connection.

The constructed class of vacuum solutions with coefficients subjected to conditions (1)–(3) is of type (28) for (29)–(31). Such metrics consist a particular case of vacuum ansatz defined by Corollary 14 with $\underline{h}_4 = 1$ and $\omega = 1$.

4.2. *Modifications of the Schwarzschild metric*

Let us consider a "prime" metric

$$ {}^{\varepsilon}\mathbf{g} = -d\xi \otimes d\xi - r^2(\xi)\ d\vartheta \otimes d\vartheta - r^2(\xi)\sin^2\vartheta\ d\varphi \otimes d\varphi + \varkappa^2(\xi)\ dt \otimes\ dt. \quad (4)$$

In general, it is not obligatory to consider modifications only of solutions of Einstein equations. Our goal is to construct a class of nonholonomic deformations into "target" off–diagonal ones generating solutions of some (effective) vacuum Einstein equations. The "primary" geometric data for (4) are stated by nontrivial coefficients

$$\check{g}_1 = -1,\ \check{g}_2 = -r^2(\xi),\ \check{h}_3 = -r^2(\xi)\sin^2\vartheta,\ \check{h}_4 = \varkappa^2(\xi), \quad (5)$$

for local coordinates $x^1 = \xi, x^2 = \vartheta, y^3 = \varphi, y^4 = t$, where

$$\xi = \int dr\ \left|1 - \frac{2\mu_0}{r} + \frac{\varepsilon}{r^2}\right|^{1/2} \text{ and } \varkappa^2(r) = 1 - \frac{2\mu_0}{r} + \frac{\varepsilon}{r^2}.$$

In a particular cas for $\varepsilon = 0$ and μ_0 considered as a point mass, the metric ${}^{\varepsilon}\mathbf{g}$ (4) determines the Schwarzschild solution.

We generate exact solutions of the system (29)–(31) with effective ${}^{v}\Upsilon = \Upsilon$ and $\Upsilon + {}^{s}\lambda = 0$ via nonholonomic deformations ${}^{\varepsilon}\mathbf{g} \to {}^{\varepsilon}_{\eta}\mathbf{g}$, when $g_i = \eta_i \check{g}_i$ and $h_a = \eta_a \check{h}_a$ and w_i, n_i. The resulting class of target metrics is parameterized in the form

$$\begin{aligned} {}^{\varepsilon}_{\eta}\mathbf{g} = & \eta_1(\xi) d\xi \otimes d\xi + \eta_2(\xi) r^2(\xi)\ d\vartheta \otimes d\vartheta + \\ & \eta_3(\xi,\vartheta,\varphi) r^2(\xi)\sin^2\vartheta\ \delta\varphi \otimes \delta\varphi - \eta_4(\xi,\vartheta,\varphi)\varpi^2(\xi)\ \delta t \otimes \delta t, \\ \delta\varphi = & d\varphi + w_1(\xi,\vartheta,\varphi)d\xi + w_2(\xi,\vartheta,\varphi)d\vartheta,\ \delta t = dt + n_1(\xi,\vartheta)d\xi + n_2(\xi,\vartheta)d\vartheta, \end{aligned} \quad (7)$$

when the modified gravitational field equations for zero effective source relate the prime and target coefficients of the vertical metric and polarization functions via formulas

$$h_3 = h_0^2(\varpi^*)^2 = \eta_3(\xi,\vartheta,\varphi) r^2(\xi)\sin^2\vartheta,\ h_4 = -\varpi^2 = -\eta_4(\xi,\vartheta,\varphi)\varkappa^2(\xi). \quad (8)$$

In these formulas, $|\eta_3| = (h_0)^2|\check{h}_4/\check{h}_3|[(\sqrt{|\eta_4|})^*]^2$ and we have to chose $h_0 = const$ ($h_0 = 2$ in order to satisfy the first condition (3)). The values $\check{h}_a$ are taken for the Schwarzschild solution for the chosen system of coordinates and η_4 can be any function with $\eta_4^* \neq 0$. The f–modified gravitational polarizations η_1 and η_2, when $\eta_1 = \eta_2 r^2 = e^{\psi(\xi,\vartheta)}$, are found from (38) with zero source, written in the form $\psi^{\bullet\bullet} + \psi'' = 0$.

Introducing the coefficients (8) in the ansatz (7), we find a class of exact off–diagonal effective vacuum solutions of the Einstein equations defining stationary nonholonomic deformations of the Schwarzschild metric,

$$
\begin{aligned}
{}^{\varepsilon}\mathbf{g} &= -e^{\psi}\,(d\xi\otimes d\xi+\ d\vartheta\otimes d\vartheta) - 4\left[(\sqrt{|\eta_4|})^{*}\right]^{2}\varkappa^{2}\ \delta\varphi\otimes\ \delta\varphi+\eta_4\varkappa^{2}\ \delta t\otimes\delta t,\\
\delta\varphi &= d\varphi+w_1 d\xi+w_2 d\vartheta,\ \delta t = dt+\ {}^{1}n_1 d\xi+\ {}^{1}n_2 d\vartheta. \qquad (9)
\end{aligned}
$$

The N–connection coefficients $w_i(\xi,\vartheta,\varphi)$ and ${}^{1}n_i(\xi,\vartheta)$ must satisfy the conditions (3) in order to get effect vacuum metrics with generic off–diagonal terms in GR. Finally, we emphasize here that, in general, the bulk of solutions from the set of target metrics do not define black holes and do not describe obvious physical situations. f–modifications may preserve the singular character of the coefficient ϖ^2 vanishing on the horizon of a Schwarzschild black hole if we take only smooth integration functions for some small deformation parameters ε.

4.3. *Linear parametric polarizations and f–modifications induced by YMH fields*

Let us select effective gravitational vacuum configurations with spherical and/or rotoid (ellipsoid) symmetry if it is considered a generating function

$$\varpi^2 = \iota(\xi,\vartheta,\varphi)+\varepsilon\varrho(\xi,\vartheta,\varphi). \qquad (10)$$

For simplicity, we shall restrict our construction only to linear decompositions on a small parameter ε, with $0<\varepsilon<<1$.

Using (10), we compute $(\varpi^*)^2 = [(\sqrt{|\iota|})^*]^2\ [1+\varepsilon\frac{1}{(\sqrt{|\iota|})^*}(\varrho/\sqrt{|\iota|})^*]$ and the vertical coefficients of d–metric (9), i.e h_3 and h_4 (and corresponding polarizations η_3 and η_4), see formulas (8). For rotoid configurations,

$$\iota = 1-\frac{2\mu(\xi,\vartheta,\varphi)}{r} \text{ and } \varrho = \frac{\iota_0(r)}{4\mu^2}\sin(\omega_0\varphi+\varphi_0), \qquad (11)$$

for $\mu(\xi,\vartheta,\varphi)=\mu_0+\varepsilon\mu_1(\xi,\vartheta,\varphi)$ (supposing that the mass is locally anisotropically polarized) with certain constants μ,ω_0 and φ_0 and arbitrary functions/ polarizations $\mu_1(\xi,\vartheta,\varphi)$ and $\iota_0(r)$ to be determined from some boundary conditions, with ε being the eccentricity. We may treat ε as an eccentricity imposing the condition that the coefficient $h_4=\varpi^2=\eta_4(\xi,\vartheta,\varphi)\varkappa^2(\xi)$ becomes zero for data (11) if

$$r_+ \simeq 2\mu_0/(1+\varepsilon\frac{\iota_0(r)}{4\mu^2}\sin(\omega_0\varphi+\varphi_0)).$$

Such conditions result in small deformations of the Schwarzschild spherical horizon into an ellipsoidal one (rotoid configuration with eccentricity ε).

The resulting target solutions are for off–diagonal solution with rotoid type symmetry

$${}^{rot}\mathbf{g} = -e^{\psi}\,(d\xi\otimes d\xi+\ d\vartheta\otimes d\vartheta)+(q+\varepsilon\varrho)\ \delta t\otimes\delta t$$

$$-4\left[(\sqrt{|\iota|})^{*}\right]^{2}\ [1+\varepsilon\frac{1}{(\sqrt{|\iota|})^{*}}(\varrho/\sqrt{|\iota|})^{*}]\ \delta\varphi\otimes\ \delta\varphi, \quad (13)$$
$$\delta\varphi = d\varphi + w_1 d\xi + w_2 d\vartheta,\ \delta t = dt + \ ^1n_1 d\xi + \ ^1n_2 d\vartheta.$$

The functions $\iota(\xi,\vartheta,\varphi)$ and $\varrho(\xi,\vartheta,\varphi)$ from (11) and the N–connection coefficients $w_i(\xi,\vartheta,\varphi)$ and $n_i = \ ^1n_i(\xi,\vartheta)$ should be to conditions of type (3),

$$w_1 w_2\left(\ln|\frac{w_1}{w_2}|\right)^{*} = w_2^{\bullet} - w_1', \quad w_i^{*}\neq 0; \quad (14)$$
$$\text{or}\ \ w_2^{\bullet} - w_1' = 0, \quad w_i^{*} = 0;\ ^1n_1'(\xi,\vartheta) - \ ^1n_2^{\bullet}(\xi,\vartheta) = 0$$

and $\psi(\xi,\vartheta)$ being any function for which $\psi^{\bullet\bullet}+\psi''=0$, if we are interested to generate Levi–Civita configurations.

Off–diagonal rotoid deformations of black hole solutions in GR are possible via f–deformations, in noncommutative gravity, by nonlinear YMH interactions and via generic off–diagonal Einstein gravitational fields. The generating functions and parameters of such solutions depend on the type of gravity model we consider.

5. Ellipsoid–Solitonic f–modifications of EYMH Configurations

It is possible to prescribe nonholonomic constraints with $^v\Upsilon = \Upsilon = \ ^0\Upsilon = const$ and $^0\Upsilon + \ ^s\lambda \neq 0$. This allows us to construct off–diagonal solutions for MGYMH systems (38)–(41) and (7) with coefficients of metric of type (2). Such metrics provide explicit examples of effective non–vacuum solutions with ansatz for metrics considered for the Corollary 13 with $\underline{h}_4 = 1$ and $\omega = 1$.

5.1. *Nonholonomic rotoid f–modifications*

Using the anholonomic frame method, we can generate a class of solutions with nontrivial cosmological constant possessing different limits (for large radial distances and small nonholonomic deformations) than the vacuum configurations considered in previous section.

Let us consider a diagonal metric of type

$$_{\lambda}^{\varepsilon}\mathbf{g} = d\xi\otimes d\xi + r^2(\xi)\ d\theta\otimes d\theta + r^2(\xi)\sin^2\theta\ d\varphi\otimes d\varphi + \ _{\lambda}\varkappa^2(\xi)\ dt\otimes\ dt, \quad (1)$$

where nontrivial metric coefficients are parametrized in the form $\check{g}_1 = 1,\ \check{g}_2 = r^2(\xi),\ \check{h}_3 = r^2(\xi)\sin^2\vartheta,\ \check{h}_4 = \ _{\lambda}\varkappa^2(\xi)$, for local coordinates $x^1=\xi, x^2=\vartheta, y^3=\varphi, y^4=t$, with $\xi = \int dr/|q(r)|^{\frac{1}{2}}$, and $_{\lambda}\varkappa^2(r) = -\sigma^2(r)q(r)$, for $q(r) = 1 - 2m(r)/r - \Lambda r^2/3$. In variables (r,θ,φ), the metric (1) is equivalent to (B.16).

The ansatz for such classes of solutions is chosen in the form

$$^{\lambda}\mathring{\mathbf{g}} = e^{\underline{\phi}(\xi,\theta)}\ (d\xi\otimes d\xi + \ d\theta\otimes d\theta) + h_3(\xi,\theta,\varphi)\ \delta\varphi\otimes\delta\varphi + h_4(\xi,\theta,\varphi)\ \delta t\otimes\ \delta t,$$
$$\delta\varphi = d\varphi + w_1(\xi,\theta,\varphi)\,d\xi + w_2(\xi,\theta,\varphi)\,d\theta,$$
$$\delta t = dt + n_1(\xi,\theta,\varphi)\,d\xi + n_2(\xi,\theta,\varphi)\,d\theta,$$

for $h_3 = -h_0^2(\varpi^*)^2 = \eta_3(\xi,\theta,\varphi)r^2(\xi)\sin^2\vartheta,\ h_4 = b^2 = \eta_4(\xi,\theta,\varphi)\ {}_\lambda\varkappa^2(\xi)$. The coefficients of this metric determine exact solutions if

$$\underline{\phi}^{\bullet\bullet}(\xi,\theta) + \underline{\phi}^{''}(\xi,\theta) = 2(\ {}^0\Upsilon + \ {}^s\lambda); \quad (2)$$
$$h_3 = \pm\frac{(\phi^*)^2}{4\ (\ {}^0\Upsilon + \ {}^s\lambda)}e^{-2\ {}^0\phi(\xi,\theta)},\ h_4 = \mp\frac{1}{4\ (\ {}^0\Upsilon + \ {}^s\lambda)}e^{2(\phi - \ {}^0\phi(\xi,\theta))};$$
$$w_i = \partial_i\phi/\phi^*;$$
$$n_i = \ {}^1n_i(\xi,\theta) + \ {}^2n_i(\xi,\theta)\int(\phi^*)^2 e^{-2(\phi - \ {}^0\phi(\xi,\vartheta))}d\varphi,$$
$$= \begin{cases} \ {}^1n_i(\xi,\theta) + \ {}^2n_i(\xi,\theta)\int e^{-4\phi}\frac{(h_4^*)^2}{h_4}d\varphi, & \text{if } n_i^* \neq 0; \\ \ {}^1n_i(\xi,\theta), & \text{if } n_i^* = 0; \end{cases}$$

for any nonzero coefficients h_a and h_a^* and arbitrary integrating functions, ${}^1n_i(\xi,\theta),\ {}^2n_i(\xi,\theta)$, and generating functions, $\phi(\xi,\theta,\varphi)$ and ${}^0\phi(\xi,\theta)$. Such values have to be determined from certain boundary conditions for a fixed system of coordinates and following additional assumptions depending on the type of f–modified theory of gravity we study.

For nonholonomic ellipsoid de Sitter configurations, we parameterize

$$\begin{aligned} {}^{rot}_{\lambda}\mathbf{g} &= -e^{\underline{\phi}(\xi,\theta)}(d\xi\otimes d\xi + \ d\theta\otimes d\theta) + (\underline{\iota} + \varepsilon\underline{\varrho})\ \delta t\otimes\delta t \\ &-h_0^2\left[(\sqrt{|\underline{\iota}|})^*\right]^2[1+\varepsilon\frac{1}{(\sqrt{|\underline{\iota}|})^*}(\underline{\varrho}/\sqrt{|\underline{\iota}|})^*]\ \delta\varphi\otimes\ \delta\varphi, \\ \delta\varphi &= d\varphi + w_1 d\xi + w_2 d\vartheta,\ \delta t = dt + n_1 d\xi + n_2 d\vartheta, \end{aligned} \quad (3)$$

where $\underline{\iota} = 1 - \frac{2\ {}^1\underline{\mu}(r,\theta,\varphi)}{r}$ and $\underline{\varrho} = \frac{\underline{\iota}_0(r)}{4\underline{\mu}_0^2}\sin(\omega_0\varphi + \varphi_0)$ are fixed for anisotropic rotoid configurations on the "smaller horizon" (when $h_4 = 0$),

$$r_+ \simeq 2\ {}^1\underline{\mu}/(1+\varepsilon\frac{\underline{\iota}_0(r)}{4\underline{\mu}_0^2}\sin(\omega_0\varphi+\varphi_0)),$$

for a corresponding $\underline{\iota}_0(r)$.

For the Levi–Civita configurations, we have to consider additional nonholonomic constraints resulting in zero torsion in order to generate solutions of the Einstein equations for the Levi–Civita connection. Following the condition (14), for $\phi^* \neq 0$, we obtain that $\phi(r,\varphi,\theta) = \ln|h_4^*/\sqrt{|h_3h_4|}|$ must be any function defined in non–explicit form from equation $2e^{2\phi}\phi = \ {}^0\Upsilon + \ {}^s\lambda$. It is possible to solve the set of constraints for the N–connection coefficients the integration functions in (2) are subjected to $w_1w_2\left(\ln|\frac{w_1}{w_2}|\right)^* = w_2^\bullet - w_1'$ for $w_i^* \neq 0$; $w_2^\bullet - w_1' = 0$ for $w_i^* = 0$; and take $n_i = \ {}^1n_i(x^k)$ for ${}^1n_1'(x^k) - \ {}^1n_2^\bullet(x^k) = 0$.

5.2. *Modifications via effective vacuum solitons*

Off–diagonal modifications in effective vacuum spacetimes can be modeled by 3–d solitonic gravitational interactions with nontrivial vertical conformal factor ω.

In this section, we suppose that there are satisfied the conditions of Corollary 14 with $\underline{h}_4 = 1$ for effective vacuum solutions. Such prime and target metrics may encode MGYMH configurations and their nonlinear wave deformations. Additional constraints for the Levi–Civita configurations may result in EYMH solutions.

5.2.1. *Solitonic waves for the conformal factor* $\omega(x^1, y^3, t)$

We consider functions $\omega = \eta(x^1, y^3, t)$, when $y^4 = t$ is a time like coordinate, determined by a solution of KdP equation [17],

$$\pm\, \eta^{**} + (\partial_t \eta + \eta\ \eta^{\bullet} + \epsilon \eta^{\bullet\bullet\bullet})^{\bullet} = 0, \tag{5}$$

with dispersion ϵ. In the dispersionless limit $\epsilon \to 0$ the solutions are independent on y^3 and transform into those given by Burgers' equation $\partial_t \eta + \eta\ \eta^{\bullet} = 0$. The conditions (7) are written in the form $\mathbf{e}_1 \eta = \eta^{\bullet} + w_1(x^i, y^3)\eta^* + n_1(x^i)\partial_t \eta = 0$. For $\eta' = 0$, we can impose the condition $w_2 = 0$ and $n_2 = 0$.

The corresponding effective vacuum solitonic f–modifications are given by

$$\begin{aligned} {}_1\mathbf{g} &= e^{\psi(x^k)}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) + \left[\eta(x^1, y^3, t)\right]^2 h_a(x^1, y^3)\ \mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dy^3 + w_1(x^k, y^3)dx^1,\ \mathbf{e}^4 = dy^4 + n_1(x^k)dx^1. \end{aligned}$$

This class of metrics depend on all spacetime coordinates and may not posses, in general, Killing symmetries. Nevertheless, there are symmetries determined by solitonic solutions of (5). Alternatively, we can consider that η is a solution of any three dimensional solitonic and/ or other nonlinear wave equations; in a similar manner, we can generate solutions for $\omega = \eta(x^2, y^3, t)$.

5.2.2. *f–modifications with solitonic factor* $\omega(x^i, t)$

There are off–diagonal solutions when the effective vacuum metrics are with a solitonic dynamics not depending on anisotropic coordinate y^3. To generate such nonholonomic configurations we take $\omega = \widehat{\eta}(x^k, t)$ is a solution of KdP equation

$$\pm\, \widehat{\eta}^{\bullet\bullet} + (\partial_t \widehat{\eta} + \widehat{\eta}\ \widehat{\eta}' + \epsilon \widehat{\eta}''')' = 0 \tag{6}$$

and consider that in the dispersionless limit $\epsilon \to 0$ the solutions are independent on x^1 and determined by Burgers' equation $\partial_t \widehat{\eta} + \widehat{\eta}\ \widehat{\eta}' = 0$.

A class of effective vacuum solitonic EYMH configurations encoding f–modifications from MGYMH interactions is given by

$$\begin{aligned} {}_2\mathbf{g} &= e^{\psi(x^k)}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) + \left[\widehat{\eta}(x^k, t)\right]^2 h_a(x^k, y^3)\ \mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dy^3 + w_1(x^k, y^3)dx^1,\ \mathbf{e}^4 = dy^4 + n_1(x^k)dx^1, \end{aligned}$$

when (7) are equivalent to $\mathbf{e}_1 \widehat{\eta} = \widehat{\eta}^{\bullet} + n_1(x^i)\partial_t \widehat{\eta} = 0,\ \mathbf{e}_2 \widehat{\eta} = \widehat{\eta}' + n_2(x^i)\partial_t \widehat{\eta} = 0.$

Finally, we note that modified gravity theories can be characterized by exact solutions with an infinite number of vacuum gravitational 2-d and 3-d configurations stated by corresponding solitonic hierarchies and bi–Hamilton structures, for instance, related to different KdP equations (6). There are possible mixtures with solutions for 2-d and 3-d sine–Gordon equations etc, see details in Ref. [18]. The constants, parametric dependence and generating functions are determined by corresponding models of modified gravity and possible extra dimension generalizations.

6. Concluding Remarks

As a consequence of the discovery of the accelerating expansion of the Universe and attempts to formulate self–consistent schemes and propose an experimentally verifiable phenomenology for quantum gravity a number of modified gravity (MG) theories have been proposed along recent years. It is considered that a change of the paradigm of standard particle theory is inevitable in order to understand and solve the dark energy and dark matter problems. In this sense, the $f(R, T, ...)$–theories with functional dependence on various types of scalar curvatures, torsions, energy–momentum tensors etc have become popular candidates which may be capable to solve various puzzles in particle physics and modern cosmology.

It is considered that viable modified gravity theories should be characterized by a well behavior at local scales when cosmological effects like inflation and late–time acceleration are reproduced. For any candidate model to a modified/generalized gravity theory, to construct exact solutions with physical importance, describing nonlinear gravitational and matter field interactions, is a technically difficult task which requests new sophisticate geometric, analytic and numerical methods. Such exact solutions present an important theoretical tool for understanding properties of gravity theories at the classical level and suggest a number of ideas how a quantum formalism has to be developed in order to include possible modifications and corrections to cosmological and related microscopic scenarios.

In this work, we have shown that the f–modified gravitational field equations and generalizations with Yang–Mills and Higgs equations (in brief, MGYMH) can be solved in very general forms using the so–called anholonomic frame deformation method, AFDM. The approach was elaborated in a series of works on geometric methods of constructing exact solutions in Einstein gravity and its (noncommutative) generalized Finsler, brane, string modifications, see reviews of results in Refs. [12,13,14,15]. One of the most important features of the AFDM is that it propose a set of geometric constructions for decoupling certain physically important systems of nonlinear partial differential equations, PD, with respect to certain classes of nonholonomic frames. More than that, the method shows how can integrate such PDE in general form, with generic off–diagonal metrics depending on all spacetime coordinates via various integration and generating functions, symmetry parameters etc.

One of the most important conclusion of our work is that using "auxiliary" connections necessary for decoupling PDEs and generating off–diagonal configurations for metrics we can mimic various classes of f–modifications. In many cases, the geometric constructions and solutions can be constrained to be interpreted in the framework of the general relativity (GR) theory. For instance, we shown how black hole solutions in GR may be deformed (with small parameter, or in certain general forms) into off–diagonal metrics if possible f–modifications and YMH interactions are taken into considerations. New classes of metrics and connections may be with certain Killing and/or solitonic symmetries or deformed into non–Killing configurations. For well–defined conditions, a subclass of such metrics can be generated to have nontrivial limits to effective vacuum solutions, or with nonhomogeneous/ anisotropic polarizations of cosmological constants and gravitational–matter interactions. This support a conservative opinion that a number of modifications which seem to be necessary in modern cosmology and for elaborating quantum gravity models can be alternatively explained by off–diagonal, parametric and/or nonholonomic interactions in GR.

Nevertheless, the AFDM was originally elaborated, and generalized, for various modified theories of gravity. It allows us to prescribe, for instance, a convenient value of the scalar curvature for an auxiliary connection (such a curvature is not fixed, in general, for the Levi–Civita connection) or certain type of generalized matter sources and locally anisotropic nonlinear polarizations of interaction constants. Choosing a convenient nonholonomic $2+2+2+....-$ splitting, we can generate off–diagonal solutions in four and extra dimensions with cosmologically observable anisotropic behavior and related, for instance, to effective renormalized theories, see reviews of results in [4,8,7,25]). The main result of this paper is that we provided explicit proofs and explicit examples that the MGYMH equations and possible effective EYMH systems can be solved and studied using "pure" geometric and analytic methods. To quantize such nonlinear classical modified gravitational and matter field systems and study possible implications in QCD physics [26,27] is a plan for our future work.

Acknowledgments: The SV work is partially supported by the Program IDEI, PN-II-ID-PCE-2011-3-0256 and performed for a visit supported by the physics department at Kocaely University (Ismit, Turkey).

Appendix A. 2+2 Splitting of Lorentz Manifolds

We provide the main results and formulas on the canonical d–connection and corresponding d–torsion and d–curvature.

Theorem 15. *In coefficient form, the distortion relations (14) are computed*

$$\Gamma^{\gamma}{}_{\alpha\beta} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} + \widehat{\mathbf{Z}}^{\gamma}{}_{\alpha\beta}, \tag{A.1}$$

where the distortion tensor $\widehat{\mathbf{Z}}^{\gamma}{}_{\alpha\beta}$ *is*

$$Z^{i}_{jk} = Z^{a}_{bc} = 0,\ Z^{a}_{jk} = -\widehat{C}^{i}_{jb} g_{ik} g^{ab} - \frac{1}{2}\Omega^{a}_{jk},\ Z^{i}_{bk} = \frac{1}{2}\Omega^{c}_{jk} g_{cb} g^{ji} - \Xi^{ih}_{jk}\ \widehat{C}^{j}_{hb},$$
$$Z^{a}_{bk} = \ {}^{+}\Xi^{ab}_{cd}\ \widehat{T}^{c}_{kb},\ Z^{i}_{kb} = \frac{1}{2}\Omega^{a}_{jk} g_{cb} g^{ji} + \Xi^{ih}_{jk}\ \widehat{C}^{j}_{hb}, \tag{A.2}$$

$$Z^a_{jb} = {}^{-}\Xi^{ad}_{cb}\ \widehat{T}^c_{jd},\ Z^i_{ab} = -\frac{g^{ij}}{2}\left[\widehat{T}^c_{ja}g_{cb} + \widehat{T}^c_{jb}g_{ca}\right],$$

for $\Xi^{ih}_{jk} = \frac{1}{2}(\delta^i_j\delta^h_k - g_{jk}g^{ih})$ *and* ${}^{\pm}\Xi^{ab}_{cd} = \frac{1}{2}(\delta^a_c\delta^b_d + g_{cd}g^{ab})$. *The nontrivial coefficients* Ω^a_{jk} *and* $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$ *are given, respectively, by formulas (6) and, see below, (A.3).*

Proof. It follows from a straightforward verification in N–adapted frames (3) and (4) that the sums of coefficients (13) and (A.2) result in the coefficients of the Levi–Civita connection $\Gamma^{\gamma}{}_{\alpha\beta}$ for a general metric parameterized as a d–metric $\mathbf{g} = [g_{ij}, g_{ab}]$ (7). □ □

Theorem 16. *The nonholonomically induced torsion* $\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}\}$ *of* $\widehat{\mathbf{D}}$ *is determined in a unique form by the metric compatibility condition,* $\widehat{\mathbf{D}}\mathbf{g} = 0$, *and zero horizontal and vertical torsion coefficients,* $\widehat{T}^i{}_{jk} = 0$ *and* $\widehat{T}^a{}_{bc} = 0$, *but with nontrivial h–v– coefficients*

$$\widehat{T}^i{}_{jk} = \widehat{L}^i_{jk} - \widehat{L}^i_{kj}, \widehat{T}^i{}_{ja} = \widehat{C}^i_{jb}, \widehat{T}^a{}_{ji} = -\Omega^a{}_{ji}, \widehat{T}^c_{aj} = \widehat{L}^c_{aj} - e_a(N^c_j), \widehat{T}^a{}_{bc} = \ \widehat{C}^a_{bc} - \ \widehat{C}^a_{cb}. \quad (A.3)$$

Proof. The coefficients (A.3) are computed by introducing $D = \widehat{\mathbf{D}}$, with coefficients (13), and $X = \mathbf{e}_\alpha, Y = \mathbf{e}_\beta$ (for N–adapted frames (3)) into standard formula for torsion, $\mathcal{T}(X,Y) := D_{\mathbf{X}}Y - D_{\mathbf{Y}}X - [X,Y]$.
□ □

In a similar form, introducing $\widehat{\mathbf{D}}$ and $X = \mathbf{e}_\alpha, Y = \mathbf{e}_\beta, Z = \mathbf{e}_\gamma$ into $\mathcal{R}(X,Y) := D_{\mathbf{X}}D_{\mathbf{Y}} - D_{\mathbf{Y}}D_{\mathbf{X}} - D_{[\mathbf{X},\mathbf{Y}]}$, we prove

Theorem 17. *The curvature* $\widehat{\mathcal{R}} = \{\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\delta}\}$ *of* $\widehat{\mathbf{D}}$ *is characterized by N–adapted coefficients*

$$\begin{aligned}
\widehat{R}^i{}_{hjk} &= e_k\widehat{L}^i{}_{hj} - e_j\widehat{L}^i{}_{hk} + \widehat{L}^m{}_{hj}\widehat{L}^i{}_{mk} - \widehat{L}^m{}_{hk}\widehat{L}^i{}_{mj} - \widehat{C}^i{}_{ha}\Omega^a{}_{kj},\\
\widehat{R}^a{}_{bjk} &= e_k\widehat{L}^a{}_{bj} - e_j\widehat{L}^a{}_{bk} + \widehat{L}^c{}_{bj}\widehat{L}^a{}_{ck} - \widehat{L}^c{}_{bk}\widehat{L}^a{}_{cj} - \widehat{C}^a{}_{bc}\Omega^c{}_{kj}, \qquad (A.4)\\
\widehat{R}^i{}_{jka} &= e_a\widehat{L}^i{}_{jk} - \widehat{D}_k\widehat{C}^i{}_{ja} + \widehat{C}^i{}_{jb}\widehat{T}^b{}_{ka}, \widehat{R}^c{}_{bka} = e_a\widehat{L}^c{}_{bk} - D_k\widehat{C}^c{}_{ba} + \widehat{C}^c{}_{bd}\widehat{T}^c{}_{ka},\\
\widehat{R}^i{}_{jbc} &= e_c\widehat{C}^i{}_{jb} - e_b\widehat{C}^i{}_{jc} + \widehat{C}^h{}_{jb}\widehat{C}^i{}_{hc} - \widehat{C}^h{}_{jc}\widehat{C}^i{}_{hb},\\
\widehat{R}^a{}_{bcd} &= e_d\widehat{C}^a{}_{bc} - e_c\widehat{C}^a{}_{bd} + \widehat{C}^e{}_{bc}\widehat{C}^a{}_{ed} - \widehat{C}^e{}_{bd}\widehat{C}^a{}_{ec}.
\end{aligned}$$

We can re–define the differential geometry of a (pseudo) Riemannian space $\mathbf{V}$ in nonholonomic form in terms of geometric data $(\mathbf{g},\widehat{\mathbf{D}})$ which is equivalent to the "standard" formulation with $(\mathbf{g}, \nabla)$.

Corollary 18. *The Ricci tensor* $\widehat{\mathbf{R}}_{\alpha\beta} := \widehat{\mathbf{R}}^{\gamma}{}_{\alpha\beta\gamma}$ *(A.9) of* $\widehat{\mathbf{D}}$ *is characterized by N–adapted coefficients*

$$\widehat{\mathbf{R}}_{\alpha\beta} = \{\widehat{R}_{ij} := \widehat{R}^k{}_{ijk},\ \widehat{R}_{ia} := -\widehat{R}^k{}_{ika},\ \widehat{R}_{ai} := \widehat{R}^b{}_{aib},\ \widehat{R}_{ab} := \widehat{R}^c{}_{abc}\}. \qquad (A.5)$$

Proof. The formulas for h–v–components (A.5) are obtained by contracting respectively the coefficients (A.4). Using $\widehat{\mathbf{D}}$ (13), we express such formulas in terms of partial derivatives of coefficients of metric $\mathbf{g}$ (1) and any equivalent parametrization in the form (7), or (8). □ □

The scalar curvature ${}^s\widehat{R}$ of $\widehat{\mathbf{D}}$ is by definition

$$ {}^s\widehat{R} := \mathbf{g}^{\alpha\beta}\widehat{\mathbf{R}}_{\alpha\beta} = g^{ij}\widehat{R}_{ij} + g^{ab}\widehat{R}_{ab}. \quad (A.6)$$

Using (A.5) and (A.6), we can compute the Einstein tensor $\widehat{\mathbf{E}}_{\alpha\beta}$ of $\widehat{\mathbf{D}}$,

$$\widehat{\mathbf{E}}_{\alpha\beta} := \widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\ {}^s\widehat{R}. \quad (A.7)$$

In general, this tensor is different from that constructed using (A.8) for the Levi–Civita connection ∇.

Proposition 19. *The N–adapted coefficients* $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ *of* $\widehat{\mathbf{D}}$ *are identic to the coefficients* $\Gamma^{\gamma}{}_{\alpha\beta}$ *of* ∇*, both sets computed with respect to N–adapted frames (3) and (4), if and only if there are satisfied the conditions* $\widehat{L}^c_{aj} = e_a(N^c_j), \widehat{C}^i_{jb} = 0$ *and* $\Omega^a{}_{ji} = 0.$

Proof. If the conditions of the Proposition, i.e. constraints (23), are satisfied, all N–adapted coefficients of the torsion $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$ (A.3) are zero. In such a case, the distortion tensor $\widehat{\mathbf{Z}}^{\gamma}{}_{\alpha\beta}$ is also zero. Following formula (A.1), we get $\Gamma^{\gamma}{}_{\alpha\beta} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$. Inversely, if the last equalities of coefficients are satisfied for a chosen splitting (2), we get trivial torsions and distortions of ∇. We emphasize that, in general, $\widehat{\mathbf{D}} \neq \nabla$ because such connections have different transformation rules under frame/coordinate transforms. Nevertheless, if $\Gamma^{\gamma}{}_{\alpha\beta} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ in a N–adapted frame of reference, we get corresponding equalities for the Riemann and Ricci tensors etc. This means that the N–coefficients are such way fixed via frame transforms that the nonholonomic distribution became integrable even, in general, the frames (3) and (4) are nonholonomic (because not all anholonomy coefficients are not obligatory zero, for instance, $w^b_{ia} = \partial_a N^b_i$ may be nontrivial, see formulas (6)). □ □

In order to elaborate models of gravity theories for ∇ and/or $\widehat{\mathbf{D}}$, we have to consider the corresponding Ricci tensors,

$$Ric = \{R\ _{\beta\gamma} := R^{\alpha}{}_{\beta\gamma\alpha}\},\ \text{for } \nabla = \{\Gamma^{\gamma}{}_{\alpha\beta}\}, \quad (A.8)$$

$$\text{and } \widehat{R}ic = \{\widehat{\mathbf{R}}\ _{\beta\gamma} := \widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\alpha}\},\ \text{for } \widehat{\mathbf{D}} = \{\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}\}. \quad (A.9)$$

Appendix B. Proof of Theorem 6

For $\omega = 1$ and $\underline{h}_a = const$, such proofs can be obtained by straightforward computations [12], see also Appendices to [15]. The approach was extended for $\omega \neq 1$ and higher dimensions in [13,14]. In this section, we sketch a proof for ansatz (9) with nontrivial $\underline{h}_4$ depending on variable y^4 when $\omega = 1$ in data (10). At the next step, the formulas will be completed for nontrivial values $\omega \neq 1$.

Using $\widehat{R}^1{}_1 = \widehat{R}^2{}_2$ and $\widehat{R}^3{}_3 = \widehat{R}^4{}_4$, the equations (20) for $\widehat{\mathbf{D}}$ and data (B.2) (see below) can be written for any source (21) in the form

$$\widehat{E}^1{}_1 = \widehat{E}^2{}_2 = -\widehat{R}^3{}_3 = \mathbf{\Upsilon}(x^k, y^3) + \underline{\mathbf{\Upsilon}}(x^k, y^3, y^4),\ \widehat{E}^3{}_3 = \widehat{E}^4{}_4 = -\widehat{R}^1{}_1 = {}^v\mathbf{\Upsilon}(x^k).$$

The geometric data for the conditions of Theorem 6 are $g_i = g_i(x^k)$ and

$$g_3 = h_3(x^k, y^3), g_4 = h_4(x^k, y^3)\underline{h}_4(x^k, y^4), N_i^3 = w_i(x^k, y^3), N_i^4 = n_i(x^k, y^3), \quad \text{(B.2)}$$

for $\underline{h}_3 = 1$ and local coordinates $u^\alpha = (x^i, y^a) = (x^1, x^2, y^3, y^4)$. For such values, we shall compute respectively the coefficients of $\Omega\ ^a_{\alpha\beta}$ in (6), canonical d–connection $\widehat{\mathbf{\Gamma}}^\gamma{}_{\alpha\beta}$ (13), d–torsion $\widehat{\mathbf{T}}^\gamma{}_{\alpha\beta}$ (A.3), necessary coefficients of d–curvature $\widehat{\mathbf{R}}^\tau{}_{\alpha\beta\gamma}$ (A.4) with respective contractions for $\widehat{\mathbf{R}}_{\alpha\beta} := \widehat{\mathbf{R}}^\gamma{}_{\alpha\beta\gamma}$ (A.5) and resulting ${}^s\widehat{R}$ (A.6) and $\widehat{\mathbf{E}}_{\alpha\beta}$(A.7). Finally, we shall state the conditions (23) when general coefficients (B.2) are considered for d–metrics.

B.1. *The coefficients of the canonical d–connection and its torsion*

There are such nontrivial coefficients of $\widehat{\mathbf{\Gamma}}^\gamma{}_{\alpha\beta}$ (13),

$$\widehat{L}^1_{11} = \frac{g_1^\bullet}{2g_1},\ \widehat{L}^1_{12} = \frac{g_1'}{2g_1}, \widehat{L}^1_{22} = -\frac{g_2^\bullet}{2g_1},\ \widehat{L}^2_{11} = \frac{-g_1'}{2g_2},\ \widehat{L}^2_{12} = \frac{g_2^\bullet}{2g_2},\ \widehat{L}^2_{22} = \frac{g_2'}{2g_2}, \quad \text{(B.3)}$$
$$\widehat{L}^4_{4k} = \frac{\partial_k(h_4\underline{h}_4)}{2h_4\underline{h}_4} - \frac{w_k h_4^*}{2h_4} - (n_k + \underline{n}_k)\frac{\underline{h}_4^\circ}{2\underline{h}_4}, \widehat{L}^3_{3k} = \frac{\partial_k h_3}{2h_3} - \frac{w_k h_3^*}{2h_3}, \widehat{L}^3_{4k} = \frac{h_4\underline{h}_4}{-2h_3}n_k^*,$$
$$\widehat{L}^4_{3k} = \frac{1}{2}n_k^*, \widehat{C}^3_{33} = \frac{h_3^*}{2h_3}, \widehat{C}^3_{44} = -\frac{h_4^*\underline{h}_4}{h_3},\ \widehat{C}^4_{33} = -\frac{h_3\underline{h}_3^\circ}{h_4\underline{h}_4},\ \widehat{C}^4_{34} = \frac{h_4^*}{2h_4}, \widehat{C}^4_{44} = \frac{\underline{h}_4^\circ}{2\underline{h}_4}.$$

We shall need also the values

$$\widehat{C}_3 = \widehat{C}^3_{33} + \widehat{C}^4_{34} = \frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}, \widehat{C}_4 = \widehat{C}^3_{43} + \widehat{C}^4_{44} = \frac{\underline{h}_4^\circ}{2\underline{h}_4}. \quad \text{(B.4)}$$

Using data (B.2) for $\underline{w}_i = \underline{n}_i = 0$, the coefficients $\Omega^a_{ij} = \mathbf{e}_j\,(N^a_i) - \mathbf{e}_i(N^a_j)$ (6), are computed

$$\Omega^a_{ij} = \partial_j\,(N^a_i) - \partial_i(N^a_j) - w_i(N^a_j)^* + w_j(N^a_i)^*.$$

There are such nontrivial values

$$\Omega^3_{12} = -\Omega^3_{21} = \partial_2 w_1 - \partial_1 w_2 - w_1 w_2^* + w_2 w_1^* = w_1' - w_2^\bullet - w_1 w_2{}^* + w_2 w_1^*;$$
$$\Omega^4_{12} = -\Omega^4_{21} = \partial_2 n_1 - \partial_1 n_2 - w_1 n_2^* + w_2 n_1^* = n_1' - n_2^\bullet - w_1 n_2^* + w_2 n_1^*. \quad \text{(B.6)}$$

The nontrivial coefficients of d–torsion (A.3) are $\widehat{T}^a{}_{ji} = -\Omega^a{}_{ji}$ (B.6) and $\widehat{T}^c_{aj} = \widehat{L}^c_{aj} - e_a(N^c_j)$. For other types of coefficients,

$$\widehat{T}^i{}_{jk} = \widehat{L}^i_{jk} - \widehat{L}^i_{kj} = 0,\ \widehat{T}^i{}_{ja} = \widehat{C}^i_{jb} = 0,\ \widehat{T}^a{}_{bc} = \widehat{C}^a_{bc} - \widehat{C}^a_{cb} = 0,$$
$$\widehat{T}^3_{3k} = \widehat{L}^3_{3k} - e_3(N^3_k) = \frac{\partial_k h_3}{2h_3} - w_k\frac{h_3^*}{2h_3} - w_k^*,$$

$$
\begin{aligned}
\widehat{T}^3_{4k} &= \widehat{L}^3_{4k} - e_4(N^3_k) = -\frac{h_4\underline{h}_4}{2h_3}n^*_k,\ \widehat{T}^4_{3k} = \ \widehat{L}^4_{3k} - e_3(N^4_k) = \frac{1}{2}n^*_k - n^*_k = -\frac{1}{2}n^*_k,\\
\widehat{T}^4_{4k} &= \widehat{L}^4_{4k} - e_4(N^4_k) = \frac{\partial_k(h_4\underline{h}_4)}{2h_4\underline{h}_4} - w_k\frac{h^*_4}{2h_4} - n_k\frac{\underline{h}^\circ_4}{2\underline{h}_4},\\
-\widehat{T}^3_{12} &= w'_1 - w^\bullet_2 - w_1w^*_2 + w_2w^*_1,\ -\widehat{T}^4_{12} = n'_1 - n^\bullet_2 - w_1n^*_2 + w_2n^*_1. \qquad \text{(B.7)}
\end{aligned}
$$

Such coefficients of torsion if and only if $\Gamma^{\gamma}{}_{\alpha\beta} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$.

B.2. *The zero torsion conditions*

We must solve the equations

$$
\widehat{T}^4_{4k} = \widehat{L}^4_{4k} - e_4(N^4_k) = \frac{\partial_k(h_4\underline{h}_4)}{2h_4\underline{h}_4} - w_k\frac{h^*_4}{2h_4} - n_k\frac{\underline{h}^\circ_4}{2\underline{h}_4} = 0,
$$

which follow from formulas (B.7). Taking any $\underline{h}_4$ for which

$$
n_k\underline{h}^\circ_4 = \partial_k\underline{h}_4, \qquad \text{(B.9)}
$$

the condition $n_k\frac{h^*_4}{2h_4}\frac{\underline{h}^\circ_4}{2\underline{h}_4} - \frac{h^*_4}{2h_4}\frac{\partial_k\underline{h}_4}{2\underline{h}_4} = 0$ can be satisfied. For instance, parametrizing $\underline{h}_4 = {}^h\underline{h}_4(x^k)\underline{h}(y^4)$, the equations (B.9) are solved by any

$$
\underline{h}(y^4) = e^{\varkappa y^4} \text{ and } n_k = \varkappa\partial_k[\ {}^h\underline{h}_4(x^k)],\ \text{for } \varkappa = const.
$$

We conclude that for any n_k and $\underline{h}_4$ related by conditions (B.9) the zero torsion conditions (B.7) are the same as for $\underline{h}_4 = const$. Using a similar proof from [13,14], it is possible to verify by straightforward computations that $\widehat{T}^\alpha_{\beta\gamma} = 0$ if the equations (29) are solved.

B.3. *N–adapted coefficients of the canonical Ricci d–tensor*

The values $\widehat{R}_{ij} = \widehat{R}^k{}_{ijk}$ are computed as (A.9) using (A.4),

$$
\begin{aligned}
\widehat{R}^i{}_{hjk} &= \mathbf{e}_k\widehat{L}^i_{.hj} - \mathbf{e}_j\widehat{L}^i_{hk} + \widehat{L}^m_{hj}\widehat{L}^i_{mk} - \widehat{L}^m_{hk}\widehat{L}^i_{mj} - \widehat{C}^i_{ha}\Omega^a_{jk}\\
&= \partial_k\widehat{L}^i_{.hj} - \partial_j\widehat{L}^i_{hk} + \widehat{L}^m_{hj}\widehat{L}^i_{mk} - \widehat{L}^m_{hk}\widehat{L}^i_{mj}.
\end{aligned}
$$

We note $\widehat{C}^i{}_{ha} = 0$ and

$$
\mathbf{e}_k\widehat{L}^i_{hj} = \partial_k\widehat{L}^i_{hj} + N^a_k\partial_a\widehat{L}^i_{hj} = \partial_k\widehat{L}^i_{hj} + w_k\left(\widehat{L}^i_{hj}\right)^* + n_k\left(\widehat{L}^i_{hj}\right)^\circ = \partial_k\widehat{L}^i_{hj}.
$$

Taking derivatives of (B.3), we obtain

$$
\begin{aligned}
&\partial_1\widehat{L}^1{}_{11} = (\frac{g^\bullet_1}{2g_1})^\bullet = \frac{g^{\bullet\bullet}_1}{2g_1} - \frac{(g^\bullet_1)^2}{2\,(g_1)^2},\ \partial_1\widehat{L}^1{}_{12} = (\frac{g'_1}{2g_1})^\bullet = \frac{g'^\bullet_1}{2g_1} - \frac{g^\bullet_1 g'_1}{2\,(g_1)^2},\\
&\partial_1\widehat{L}^1{}_{22} = (-\frac{g^\bullet_2}{2g_1})^\bullet = -\frac{g^{\bullet\bullet}_2}{2g_1} + \frac{g^\bullet_1 g^\bullet_2}{2\,(g_1)^2},\ \partial_1\widehat{L}^2{}_{11} = (-\frac{g'_1}{2g_2})^\bullet = -\frac{g'^\bullet_1}{2g_2} + \frac{g^\bullet_1 g'_2}{2\,(g_2)^2},\\
&\partial_1\widehat{L}^2{}_{12} = (\frac{g^\bullet_2}{2g_2})^\bullet = \frac{g^{\bullet\bullet}_2}{2g_2} - \frac{(g^\bullet_2)^2}{2\,(g_2)^2},\ \partial_1\widehat{L}^2{}_{22} = (\frac{g'_2}{2g_2})^\bullet = \frac{g'^\bullet_2}{2g_2} - \frac{g^\bullet_2 g'_2}{2\,(g_2)^2},
\end{aligned}
$$

$$\partial_2\widehat{L}^1{}_{11} = (\frac{g_1^\bullet}{2g_1})' = \frac{g_1^{\bullet\prime}}{2g_1} - \frac{g_1^\bullet g_1'}{2\,(g_1)^2},\ \partial_2\widehat{L}^1{}_{12} = (\frac{g_1'}{2g_1})' = \frac{g_1''}{2g_1} - \frac{(g_1')^2}{2\,(g_1)^2},$$
$$\partial_2\widehat{L}^1{}_{22} = (-\frac{g_2^\bullet}{2g_1})' = -\frac{g_2^{\bullet\prime}}{2g_1} + \frac{g_2^\bullet g_1'}{2\,(g_1)^2},\ \partial_2\widehat{L}^2{}_{11} = (-\frac{g_1'}{2g_2})' = -\frac{g_1''}{2g_2} + \frac{g_1^\bullet g_1'}{2\,(g_2)^2},$$
$$\partial_2\widehat{L}^2{}_{12} = (\frac{g_2^\bullet}{2g_2})' = \frac{g_2^{\bullet\prime}}{2g_2} - \frac{g_2^\bullet g_2'}{2\,(g_2)^2}, \partial_2\widehat{L}^2{}_{22} = (\frac{g_2'}{2g_2})' = \frac{g_2''}{2g_2} - \frac{(g_2')^2}{2\,(g_2)^2}.$$

These values result in two nontrivial components,

$$\widehat{R}^1{}_{212} = \frac{g_2^{\bullet\bullet}}{2g_1} - \frac{g_1^\bullet g_2^\bullet}{4\,(g_1)^2} - \frac{(g_2^\bullet)^2}{4g_1g_2} + \frac{g_1''}{2g_1} - \frac{g_1'g_2'}{4g_1g_2} - \frac{(g_1')^2}{4\,(g_1)^2},$$
$$\widehat{R}^2{}_{112} = -\frac{g_2^{\bullet\bullet}}{2g_2} + \frac{g_1^\bullet g_2^\bullet}{4g_1g_2} + \frac{(g_2^\bullet)^2}{4(g_2)^2} - \frac{g_1''}{2g_2} + \frac{g_1'g_2'}{4(g_2)^2} + \frac{(g_1')^2}{4g_1g_2}.$$

Considering $\widehat{R}_{11} = -\widehat{R}^2{}_{112}$ and $\widehat{R}_{22} = \widehat{R}^1{}_{212}$, for $g^i = 1/g_i$, we compute

$$\widehat{R}^1_1 = \widehat{R}^2_2 = -\frac{1}{2g_1g_2}[g_2^{\bullet\bullet} - \frac{g_1^\bullet g_2^\bullet}{2g_1} - \frac{(g_2^\bullet)^2}{2g_2} + g_1'' - \frac{g_1'g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}],$$

which is contained in equations (39).

To derive the equations (40) we consider the third formula in (A.4),

$$\widehat{R}^c{}_{bka} = \frac{\partial \widehat{L}^c_{bk}}{\partial y^a} - \widehat{C}^c{}_{ba|k} + \widehat{C}^c{}_{bd}\widehat{T}^d_{ka}$$
$$= \frac{\partial \widehat{L}^c_{bk}}{\partial y^a} - (\frac{\partial \widehat{C}^c_{ba}}{\partial x^k} + \widehat{L}^c_{dk}\widehat{C}^d_{ba} - \widehat{L}^d_{bk}\widehat{C}^c_{da} - \widehat{L}^d_{ak}\widehat{C}^c_{bd}) + \widehat{C}^c_{bd}\widehat{T}^d_{ka}.$$

Contracting indices, we obtain $\widehat{R}_{bk} = \widehat{R}^a{}_{bka} = \frac{\partial L^a_{bk}}{\partial y^a} - \widehat{C}^a_{ba|k} + \widehat{C}^a_{bd}\widehat{T}^d_{ka}$, where for $\widehat{C}_b := \widehat{C}^c_{ba}$

$$\widehat{C}_{b|k} = \mathbf{e}_k\widehat{C}_b - \widehat{L}^d{}_{bk}\widehat{C}_d = \partial_k\widehat{C}_b - N^e_k\partial_e\widehat{C}_b - \widehat{L}^d{}_{bk}\widehat{C}_d$$
$$= \partial_k\widehat{C}_b - w_k\widehat{C}^*_b - n_k\widehat{C}^\circ_b - \widehat{L}^d{}_{bk}\widehat{C}_d.$$

We consider a conventional splitting $\widehat{R}_{bk} = {}_{[1]}R_{bk} + {}_{[2]}R_{bk} + {}_{[3]}R_{bk}$, where

$${}_{[1]}R_{bk} = \left(\widehat{L}^3_{bk}\right)^* + \left(\widehat{L}^4_{bk}\right)^\circ,\ {}_{[2]}R_{bk} = -\partial_k\widehat{C}_b + w_k\widehat{C}^*_b + n_k\widehat{C}^\circ_b + \widehat{L}^d{}_{bk}\widehat{C}_d,$$
$${}_{[3]}R_{bk} = \widehat{C}^a_{bd}\widehat{T}^d_{ka} = \widehat{C}^3_{b3}\widehat{T}^3_{k3} + \widehat{C}^3_{b4}\widehat{T}^4_{k3} + \widehat{C}^4_{b3}\widehat{T}^3_{k4} + \widehat{C}^4_{b4}\widehat{T}^4_{k4}.$$

Using formulas (B.3), (B.7) and (B.4), we compute

$${}_{[1]}R_{3k} = \left(\widehat{L}^3_{3k}\right)^* + \left(\widehat{L}^4_{3k}\right)^\circ = \left(\frac{\partial_k h_3}{2h_3} - w_k\frac{h_3^*}{2h_3}\right)^*$$
$$= -w_k^*\frac{h_3^*}{2h_3} - w_k\left(\frac{h_3^*}{2h_3}\right)^* + \frac{1}{2}\left(\frac{\partial_k h_3}{h_3}\right)^*,$$
$${}_{[2]}R_{3k} = -\partial_k\widehat{C}_3 + w_k\widehat{C}^*_3 + n_k\widehat{C}^\circ_3 + \widehat{L}^3{}_{3k}\widehat{C}_3 + \widehat{L}^4{}_{3k}\widehat{C}_4 =$$
$$= w_k[\frac{h_3^{**}}{2h_3} - \frac{3}{4}\frac{(h_3^*)^2}{(h_3)^2} + \frac{h_4^{**}}{2h_4} - \frac{1}{2}\frac{(h_4^*)^2}{(h_4)^2} - \frac{1}{4}\frac{h_3^*}{h_3}\frac{h_4^*}{h_4}] + n_k^*\frac{\underline{h}_4^\circ}{4\underline{h}_4}$$

$$+\left(\frac{\partial_k h_3}{2h_3}+\frac{\partial_k \underline{h}_3}{2\underline{h}_3}\right)(\frac{h_3^*}{2h_3}+\frac{h_4^*}{2h_4})-\frac{1}{2}\partial_k(\frac{h_3^*}{h_3}+\frac{h_4^*}{h_4}),$$

$$_{[3]}R_{3k}=\widehat{C}_{33}^3\widehat{T}_{k3}^3+\widehat{C}_{34}^3\widehat{T}_{k3}^4+\widehat{C}_{33}^4\widehat{T}_{k4}^3+\widehat{C}_{34}^4\widehat{T}_{k4}^4=w_k\left(\frac{(h_3^*)^2}{4(h_3)^2}+\frac{(h_4^*)^2}{4(h_4)^2}\right)$$
$$+w_k^*\frac{h_3^*}{2h_3}+n_k\frac{h_4^*}{2h_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}-\frac{h_3^*}{2h_3}\frac{\partial_k h_3}{2h_3}-\frac{h_4^*}{2h_4}\left(\frac{\partial_k h_4}{2h_4}+\frac{\partial_k \underline{h}_4}{2\underline{h}_4}\right).$$

Putting together, we get

$$\widehat{R}_{3k}=w_k\left[\frac{h_4^{**}}{2h_4}-\frac{1}{4}\frac{(h_4^*)^2}{(h_4)^2}-\frac{1}{4}\frac{h_3^*}{h_3}\frac{h_4^*}{h_4}\right]+n_k^*\frac{\underline{h}_4^\circ}{4\underline{h}_4}+n_k\frac{h_4^*}{2h_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}$$
$$+\frac{h_4^*}{2h_4}\frac{\partial_k h_3}{2h_3}-\frac{1}{2}\frac{\partial_k h_4^*}{h_4}+\frac{1}{4}\frac{h_4^*\partial_k h_4}{(h_4)^2}-\frac{h_4^*}{2h_4}\frac{\partial_k \underline{h}_4}{2\underline{h}_4},$$

which is equivalent to (40) if the conditions $n_k\underline{h}_4^\circ=\partial_k\underline{h}_4$, see below formula (B.9), are satisfied.

The values $\widehat{R}_{4k}={}_{[1]}R_{4k}+{}_{[2]}R_{4k}+{}_{[3]}R_{4k}$, are defined by

$$_{[1]}R_{4k}=\left(\widehat{L}_{4k}^3\right)^*+\left(\widehat{L}_{4k}^4\right)^\circ,\quad {}_{[2]}R_{4k}=-\partial_k\widehat{C}_4+w_k\widehat{C}_4^*+n_k\widehat{C}_4^\circ+\widehat{L}^3{}_{4k}\widehat{C}_3$$
$$+\widehat{L}^4{}_{4k}\widehat{C}_4,\quad {}_{[3]}R_{4k}=\widehat{C}_{4d}^a\widehat{T}_{ka}^d=\widehat{C}_{43}^3\widehat{T}_{k3}^3+\widehat{C}_{44}^3\widehat{T}_{k3}^4+\widehat{C}_{43}^4\widehat{T}_{k4}^3+\widehat{C}_{44}^4\widehat{T}_{k4}^4.$$

Using $\widehat{L}_{4k}^3$ and $\widehat{L}_{4k}^4$ from (B.3), we obtain

$$_{[1]}R_{4k}=\left(\widehat{L}_{4k}^3\right)^*+\left(\widehat{L}_{4k}^4\right)^\circ=-(\frac{h_4\underline{h}_4}{2h_3}n_k^*)^*+(\frac{\partial_k(h_4\underline{h}_4)}{2h_4\underline{h}_4}-w_k\frac{h_4^*}{2h_4}-n_k)\frac{\underline{h}_4^\circ}{2\underline{h}_4})^\circ$$
$$=-n_k^{**}\frac{h_4}{2h_3}\underline{h}_4-n_k^*(\frac{h_4^*}{2h_3}-\frac{h_4^*h_3^*}{2(h_3)^*})\underline{h}_4-n_k(\frac{\underline{h}_4^{\circ\circ}}{2\underline{h}_4}-\frac{(\underline{h}_4^\circ)^2}{2(\underline{h}_4)^2})+\frac{\partial_k\underline{h}_4^\circ}{2\underline{h}_4}-\frac{\underline{h}_4^\circ\partial_k\underline{h}_4}{2(\underline{h}_4)^2}.$$

The second term follows from (B.4), for $\widehat{C}_3,\widehat{C}_4$, and (B.3), for $\widehat{L}_{4k}^3$ and $\widehat{L}_{4k}^4$,

$$_{[2]}R_{4k}=-\partial_k\widehat{C}_4+w_k\widehat{C}_4^*+n_k\widehat{C}_4^\circ+\widehat{L}^3{}_{4k}\widehat{C}_3+\widehat{L}^4{}_{4k}\widehat{C}_4$$
$$=-w_k\left(\frac{h_4^*}{2h_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}\right)-n_k^*\frac{h_4\underline{h}_4}{2h_3\underline{h}_3}\left(\frac{h_3^*}{2h_3}+\frac{h_4^*}{2h_4}\right)$$
$$+n_k\left[\left(\frac{\underline{h}_4^\circ}{2\underline{h}_4}\right)^\circ-\frac{\underline{h}_4^\circ}{2\underline{h}_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}\right]+\frac{\partial_k h_4}{2h_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}+\frac{\partial_k\underline{h}_4}{2\underline{h}_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}-\frac{\partial_k\underline{h}_4^\circ}{2\underline{h}_4}+\frac{\underline{h}_4^\circ\partial_k\underline{h}_4}{2(\underline{h}_4)^2}.$$

The formulas (B.3), with $\widehat{C}_{43}^3,\widehat{C}_{44}^3,\widehat{C}_{43}^4,\widehat{C}_{44}^4$, and the formulas (B.7), with $\widehat{T}_{k3}^3,\widehat{T}_{k3}^4,\widehat{T}_{k4}^3,\widehat{T}_{k4}^4$, have to be used for the third term,

$$_{[3]}R_{4k}=\widehat{C}_{43}^3\widehat{T}_{k3}^3+\widehat{C}_{44}^3\widehat{T}_{k3}^4+\widehat{C}_{43}^4\widehat{T}_{k4}^3+\widehat{C}_{44}^4\widehat{T}_{k4}^4$$
$$=w_k\frac{h_4^*}{2h_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}+n_k(\frac{\underline{h}_4^\circ}{2\underline{h}_4})^2-\frac{\partial_k h_4}{2h_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}-\frac{\underline{h}_4^\circ}{2\underline{h}_4}\frac{\partial_k\underline{h}_4}{2\underline{h}_4}.$$

Summarizing above three terms,

$$\widehat{R}_{4k}=-n_k^{**}\frac{h_4}{2h_3}\underline{h}_4+n_k^*\left(-\frac{h_4^*}{2h_3}+\frac{h_4^*h_3^*}{2(h_3)^*}-\frac{h_4^*h_3^*}{4(h_3)^*}-\frac{h_4^*}{4h_3}\right)\underline{h}_4$$
$$+n_k\left(-\frac{\underline{h}_4^{\circ\circ}}{2\underline{h}_4}+\frac{(\underline{h}_4^\circ)^2}{2(\underline{h}_4)^2}+\left(\frac{\underline{h}_4^\circ}{2\underline{h}_4}\right)^\circ-\frac{\underline{h}_4^\circ}{2\underline{h}_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}+(\frac{\underline{h}_4^\circ}{2\underline{h}_4})^2\right)+\frac{\partial_k\underline{h}_4^\circ}{2\underline{h}_4}-\frac{\underline{h}_4^\circ\partial_k\underline{h}_4}{2(\underline{h}_4)^2}$$

$$+\frac{\partial_k h_4}{2h_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}+\frac{\partial_k \underline{h}_4}{2\underline{h}_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}-\frac{\partial_k \underline{h}_4^\circ}{2\underline{h}_4}+\frac{\underline{h}_4^\circ\partial_k \underline{h}_4}{2(\underline{h}_4)^2}-\frac{\partial_k h_4}{2h_4}\frac{\underline{h}_4^\circ}{2\underline{h}_4}-\frac{\underline{h}_4^\circ}{2\underline{h}_4}\frac{\partial_k \underline{h}_4}{2\underline{h}_4},$$

and prove equations (41).

For the coefficients

$$\widehat{R}^i{}_{jka}=\frac{\partial \widehat{L}^i_{jk}}{\partial y^k}-(\frac{\partial \widehat{C}^i_{ja}}{\partial x^k}+\widehat{L}^i_{lk}\widehat{C}^l_{ja}-\widehat{L}^l_{jk}\widehat{C}^i_{la}-\widehat{L}^c_{ak}\widehat{C}^i_{jc})+\widehat{C}^i_{jb}\widehat{T}^b_{ka}$$

from (A.4), we obtain zero values because $\widehat{C}^i_{jb}=0$ and $\widehat{L}^i_{jk}$ do not depend on y^k. We obtain $\widehat{R}_{ja}=\widehat{R}^i{}_{jia}=0$.

Taking $\widehat{R}^a_{\ bcd}$ from (A.4) and contracting the indices in order to compute the Ricci coefficients,

$$\widehat{R}_{bc}=\frac{\partial \widehat{C}^d_{bc}}{\partial y^d}-\frac{\partial \widehat{C}^d_{bd}}{\partial y^c}+\widehat{C}^e_{bc}\widehat{C}_e-\widehat{C}^e_{bd}\widehat{C}^d_{ec}.$$

We have

$$\widehat{R}_{bc}=(\widehat{C}^3_{bc})^*+(\widehat{C}^4_{bc})^\circ-\partial_c\widehat{C}_b+\widehat{C}^3_{bc}\widehat{C}_3+\widehat{C}^4_{bc}\widehat{C}_4-\widehat{C}^3_{b3}\widehat{C}^3_{3c}-\widehat{C}^3_{b4}\widehat{C}^4_{3c}-\widehat{C}^4_{b3}\widehat{C}^3_{4c}-\widehat{C}^4_{b4}\widehat{C}^4_{4c}.$$

There are nontrivial values,

$$\begin{aligned}\widehat{R}_{33}&=\left(\widehat{C}^3_{33}\right)^*+\left(\widehat{C}^4_{33}\right)^\circ-\widehat{C}^*_3+\widehat{C}^3_{33}\widehat{C}_3+\widehat{C}^4_{33}\widehat{C}_4-\widehat{C}^3_{33}\widehat{C}^3_{33}-2\widehat{C}^3_{34}\widehat{C}^4_{33}-\widehat{C}^4_{34}\widehat{C}^4_{43}\\&=-\frac{1}{2}\frac{h_4^{**}}{h_4}+\frac{1}{4}\frac{(h_4^*)^2}{(h_4)^2}+\frac{1}{4}\frac{h_3^*}{h_3}\frac{h_4^*}{h_4},\\ \widehat{R}_{44}&=\left(\widehat{C}^3_{44}\right)^*+\left(\widehat{C}^4_{44}\right)^\circ-\partial_4\widehat{C}_4+\widehat{C}^3_{44}\widehat{C}_3+\widehat{C}^4_{44}\widehat{C}_4-\widehat{C}^3_{43}\widehat{C}^3_{34}-2\widehat{C}^3_{44}\widehat{C}^4_{34}-\widehat{C}^4_{44}\widehat{C}^4_{44}\\&=-\frac{1}{2}\frac{h_4^{**}}{h_3}\underline{h}_4+\frac{1}{4}\frac{h_3^*h_4^*}{(h_3)^2}\underline{h}_4+\frac{1}{4}\frac{h_4^*}{h_3}\frac{h_4^*}{h_4}\underline{h}_4.\end{aligned}$$

These formulas are equivalent to nontrivial v–coefficients of the Ricci d–tensor,

$$\begin{aligned}\widehat{R}^3{}_3&=\frac{1}{h_3\underline{h}_3}\widehat{R}_{33}=\frac{1}{2h_3h_4}[-h_4^{**}+\frac{(h_4^*)^2}{2h_4}+\frac{h_3^*h_4^*}{2h_3}]\frac{1}{\underline{h}_3},\\ \widehat{R}^4{}_4&=\frac{1}{h_4\underline{h}_4}\widehat{R}_{44}=\frac{1}{2h_3h_4}[-h_4^{**}+\frac{(h_4^*)^2}{2h_4}+\frac{h_3^*h_4^*}{2h_3}]\frac{1}{\underline{h}_3},\end{aligned}$$

i.e. to the equations (39).

B.4. *Geometric data for diagonal MGYMH configurations*

The diagonal ansatz for generating solutions of the system (45)–(47) is fixed in the form

$$\begin{aligned}{}^\circ\mathbf{g}=&\ {}^\circ g_i(x^1)dx^i\otimes dx^i+\ {}^\circ h_a(x^1,x^2)dy^a\otimes dy^a= \qquad \text{(B.16)}\\ &=q^{-1}(r)dr\otimes dr+r^2d\theta\otimes d\theta+r^2\sin^2\theta d\varphi\otimes d\varphi-\sigma^2(r)q(r)dt\otimes dt,\end{aligned}$$

where the coordinates and metric coefficients are parameterized, respectively,

$$\begin{aligned}&u^\alpha=(x^1=r,x^2=\theta,y^3=\varphi,y^4=t),\\&{}^\circ g_1=q^{-1}(r),\ {}^\circ g_2=r^2,\ {}^\circ h_3=r^2\sin^2\theta,\ {}^\circ h_4=-\sigma^2(r)q(r)\end{aligned}$$

for $q(r) = 1 - 2m(r)/r - \overline{\Lambda} r^2/3$, where $\overline{\Lambda}$ is a cosmological constant. The function $m(r)$ is interpreted as the total mass–energy within the radius r which for $m(r) = 0$ defines an empty de Sitter, dS, space written in a static coordinate system with a cosmological horizon at $r = r_c = \sqrt{\frac{3}{\Lambda}}$. The solution of Yang–Mills equations (45) associated to the quadratic metric element (B.16) is defined by a single magnetic potential $\omega(r)$,

$$^{\circ}A = \ ^{\circ}A_2 dx^2 + \ ^{\circ}A_3 dy^3 = \frac{1}{2e}\left[\omega(r)\tau_1 d\theta + (\cos\theta\ \tau_3 + \omega(r)\tau_2 \sin\theta)\ d\varphi\right], \quad \text{(B.17)}$$

where τ_1, τ_2, τ_3 are Pauli matrices. The corresponding solution of (47) is given by

$$\Phi = \ ^{\circ}\Phi = \varpi(r)\tau_3. \quad \text{(B.18)}$$

Explicit values for the functions $\sigma(r), q(r), \omega(r), \varpi(r)$ have been found, for instance, in Ref. [23] following certain considerations that the data (B.16), (B.17) and (B.18), i.e. $\left[\ ^{\circ}\mathbf{g}(r),\ ^{\circ}A(r),\ \ ^{\circ}\Phi(r)\right]$, define physical solutions with diagonal metrics depending only on radial coordinate. A well known diagonal Schwarzschild–de Sitter solution of (45)–(47) is that given by data

$$\omega(r) = \pm 1, \sigma(r) = 1, \phi(r) = 0, \varkappa(r) = 1 - 2M/r - \overline{\Lambda} r^2/3$$

which defines a black hole configuration inside a cosmological horizon because $q(r) = 0$ has two positive solutions and $M < 1/3\sqrt{\overline{\Lambda}}$.

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